A link between quantum logic and categorical quantum mechanics

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Overview

Aim To provide a link between the Birkhoff and von Neumann quantum logic approach to the foundations of quantum mechanics and recent categorical approaches of Abramsky and Coecke, etc.

Specifically For each object A in a dagger biproduct symmetric monoidal category $\mathcal{C}$, we build an orthoalgebra $Proj\ A$. Certain morphisms $s : I \to A$ from the tensor unit yield finitely additive maps, or states, $\sigma_s : Proj\ A \to [0,1]_{\mathcal{C}}$ into the unit interval of scalars in $\mathcal{C}$. The tensor of the category yields a type of tensor product of orthoalgebras $Proj\ A \otimes Proj\ B$. This tensor has some, but not all, of the requirements usually sought of a tensor product in quantum logic. Several examples are explored in detail.

Definitions

Definition An OML is an ortholattice $(L, \wedge, \vee, 0, 1,')$ where $x \perp y$ implies x, y lie in a Boolean subalgebra.

Definition An OMP is an orthoposet $(P, \leq,', 0, 1)$ where $x \perp y$ implies x, y have a join and meet and lie in a Boolean subalgebra.

Definition An OA is a structure $(X, \perp, \oplus, 0, 1)$ where $\oplus$ is a partial operation with domain $\perp$ that is symmetric, associative, and satisfies

1. If $x \oplus x$ is defined then $x = 0$,
2. For each x there is a unique x' with $x \oplus x' = 1$.

Proposition OML $\Rightarrow$ OMP $\Rightarrow$ OA (where $\oplus$ is orthogonal join).

Dagger Biproduct Symmetric Monoidal Categories

Definition A DBSM-category is a category $\mathcal{C}$ with biproducts $\oplus$ equipped with a dagger $\dagger$ and monoidal structure $\otimes$ such that

1. $\pi_i^\dagger = \mu_i$
2. $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$
3. $\alpha_{A,B,C}$, $\sigma_{A,B}$, λ_A, ρ_A are unitary
4. $f \otimes (g + g') = (f \otimes g) + (f \otimes g')$
5. $(f + f') \otimes g = (f \otimes g) + (f' \otimes g)$
6. $f \otimes 0 = 0$ and $0 \otimes g = 0$.

Note The π_i, μ_i are biproduct projections and injections, $\alpha, \sigma, \lambda, \rho$ are monoidal isomorpmisms, $+$ is the additive structure from biproducts.

Proposition Strongly compact closed categories with biproducts are examples of DBSM-categories.

Weak Projections

Definition For an object A in a DBSM-category, $p : A \to A$ is a weak projection of A if there is $p' : A \to A$ where

1. p, p' are idempotent and self-adjoint
2. $pp' = 0 = p'p$
3. $p + p' = 1$

Note The p' can be shown to be unique.

Definition Let $Proj_w A$ be the set of all weak projections of A, and define a relation $\leq_w$ on this set by $p \leq_w q$ iff $pq = p = qp$.

Theorem $(Proj_w A, \leq_w, 0, 1, ')$ is an orthomodular poset. Further, if $p \perp q$, then $p \vee q = p + q$.

Remark It is well known that the idempotents of a ring with unit form an orthomodular poset where $p' = 1 - p$ and $p \leq q$ iff $pq = p = qp$. The above result shows this holds in a weaker algebraic setting than a ring.

Remark The above result doesn't use the monoidal structure $\otimes$ and the dagger structure is something tolerated by it, not something essential to it. The result would hold in any biproduct category.

Projections

Definition $p : A \to A$ is a projection if there is a biproduct $A_1 \oplus A_2$ and a unitary $u : A \to A_1 \oplus A_2$ with $p = u^\dagger \mu_1 \pi_1 u$

$$A \xrightarrow{u} A_1 \oplus A_2 \xrightarrow{\pi_1} A_1 \xrightarrow{\mu_1} A_1 \oplus A_2 \xrightarrow{u^\dagger} A$$

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Definition For projections p, q write $p \perp q$ if there is a unitary $u : A \to A_1 \oplus A_2 \oplus A_3$ with $p = u^\dagger \mu_1 \pi_1 u$ and $q = u^\dagger \mu_3 \pi_3 u$.

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Theorem $(\mathit{Proj}\ A, \perp, +, 0, 1)$ is an OA.

Theorem $\mathit{Proj}\ A$ is a sub-OA of the OMP $\mathit{Proj}_w A$. The two agree if self-adjoint idempotents strongly split.

Remark The above result does not use the monoidal structure $\otimes$ and holds in any dagger biproduct category.

Remark This result holds in a much more general setting. In many categories, the binary direct product decompositions $A \rightarrow A_1 \times A_2$ of an object A form an OA $Fact\ A$. Again, two decompositions are orthogonal when they have a common ternary refinement.

Scalars

Definition Scalars are endomorphisms $s : I \to I$ of the tensor unit. A scalar s is positive if $s = \alpha^\dagger \alpha$ for some $\alpha : I \to A$. Set $s \leq t$ if $t = s + p$ for some positive p.

Theorem The scalars form a quasiordered commutative semiring with involution.

Definition $[0,1]_{\mathcal{C}} = \{s : s \text{ is a scalar and } 0 \leq s \leq 1\}$.

States

In quantum logic, a state $\sigma : P \to [0, 1]$ is a finitely additive map from an OA to the real unit interval. Gleason's theorem says countably additve states on $L(\mathcal{H})$ correspond to density operators.

Definition $\psi : I \to A$ is a normal element if $\psi^\dagger \psi = 1$.

Theorem For $\psi : I \to A$ normal, the map $\sigma_\psi : \mathit{Proj}\ A \to [0, 1]_C$ defined by $\sigma_\psi(p) = \psi^\dagger p \psi$ for each projection $p : A \to A$ satisfies

1. $\sigma_\psi(0) = 0$
2. $\sigma_\psi(1) = 1$
3. If $p \perp q$ then $\sigma_\psi(p + q) = \sigma_\psi(p) + \sigma_\psi(q)$

Thus each normal $\psi : I \to A$ gives a state on ProjA.

Remarks It would be interesting to find conditions on $\mathcal{C}$ to ensure each A has enough normal morphisms to separate points of $Proj\ A$. It would also be of interest to know if there is a Gleason-type theorem relating states on $Proj\ A$ to normal morphisms as with $\mathcal{H}$.

Bilinear maps

Key to the notion of tensor products of OAs is the notion of bilinear maps. Roughly, these are maps that are additive in each coordinate as in linear algebra.

Definition For OAs A, B, C a map $f : A \times B \to C$ is bilinear if

1. $a_1 \perp a_2 \Rightarrow f(a_1 \oplus a_2, b) = f(a_1, b) \oplus f(a_2, b)$
2. $b_1 \perp b_2 \Rightarrow f(a, b_1 \oplus b_2) = f(a, b_1) \oplus f(a, b_2)$
3. $f(1, 1) = 1$

Tensor Products

Tensor products are an Achilles heel of quantum logic. Even the definition is problematic. Conditions 1,2 are the minimum, 3 is reasonable, 4,5 more stringent.

Definition For OAs A, B, C and $f : A \times B \to C$ consider

1. f is bilinear
2. σ, τ states on $A, B \Rightarrow \exists$ state ω on C with $\omega(f(a, b)) = \sigma(a)\tau(b)$
3. States on C are determined by their value on the image of f
4. C is generated as an OA by the image of f
5. The universal mapping property for f ala classical algebra.

Tensor Products

Theorem $\otimes : Proj\ A \times Proj\ B \to Proj\ A \otimes B$ is a bimorphism.

Theorem For states σ, τ induced by normal elements, there is a state χ with $\chi(p \otimes q) = \sigma(p)\tau(q)$.

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Remark $Proj\ A \otimes B$ has some of the proerties one asks of a tensor porduct of OAs. But

- It is not generated by the images of $Proj\ A$, $Proj\ B$.
- It does not have the universal mapping property.
- Lifting of states applies only for ones from normal elements.

Examples

FdHilb Objects are finite-dimensional Hilbert spaces, morphisms are linear maps, $\dagger$ is adjoint, $\otimes$ and $\oplus$ are usual tensor product and sum. Here all things work as expected. *Proj* $\mathcal{H}$ is the OML of closed subspaces of $\mathcal{H}$.

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Rel Objects are sets, morphisms are relations, $\dagger$ is converse, $\otimes$ is Cartesian product, $\oplus$ is disjoint union. Classical behavior with *Proj* X being the power set of X, and *Proj* $A \otimes$ *Proj* B being the usual tensor product of finite Boolean algebras.

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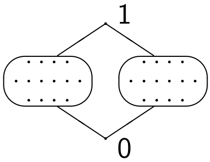
Mat_K Objects are natural numbers, morphisms from m to n are $m \times n$ matrices over the field K, $\dagger$ is transpose, $\otimes$ and $\oplus$ are multiplication and addition. Very interesting behavior ...

In Mat_K one might expect *Proj A* to be a modular ortholattice. This is not the case. Working over the field $K = \mathbb{Z}_2$ ...

- *Proj* 1, *Proj* 2, *Proj* 3 are 2, 4, 8-element Boolean algebras
- *Proj* 4 is the sum of two 16-element Boolean algebras.



- *Proj* 5 consists of six 32-element Boolean algebras pasted so that any two intersect in an atom and coatom.

Proj 4 is not modular, *Proj* 5 is not even an OMP.

Examples

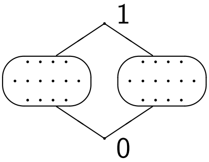
Remark The characteristic of the field K reflects itself at the level of $Proj\ A$. If $Char\ K \neq 2$ then idempotents strongly split in Mat_K, so $Proj\ A$ is always an OMP.

Lets consider tensor products in Mat_K, again with $K = \mathbb{Z}_2$

Proj $2 \otimes$ *Proj* 2

This is the tensor product of two 4-element Boolean algebras. Classically their tensor product is a 16-element Boolean algebra.

However, in Mat_K, their tensor product is *Proj* 4, a sum of two 16-element Boolean algebras.



This tensor product is exactly the embedding of the classical one, but equipped with a phantom 16-element Boolean algebra not connected to either subsystem.

Remark This shows conditions 3, 4, 5 of tensor products fail.

Concluding Remarks

- Much of what is done here likely lifts to more general settings. Biproducts may be replaced with ordinary products. The dagger seems not to be essential.
- Large parts of quantum logic deal with relating behavior in the infinite-dimensional setting with well-understood notions from projective geometry in the finite-dimensional setting.
- There may be something to learn from quantum logic in extending and refining the categorical approach.

Many thanks to the organizers.

Thank you for listening.

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