Boolean Subalgebras of Orthomodular Posets

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Overview

For an orthomodular poset A we consider BSub(A), its poset of Boolean subalgebras. We do the following.

1. Reconstruct A from BSub(A) via "directions"
2. Characterize posets arising as BSub(A)
3. Give a near categorical equivalence OMPs $\simeq$ such posets
4. Give a graphical tool to work with such posets
5. Give connections to projective geometry
6. Discuss applications in the topos approach to QM

Projective geometry

Let V be a vector space over a field F.

Definition Sub(V) is the lattice of subspaces of V.

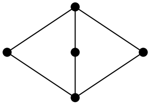
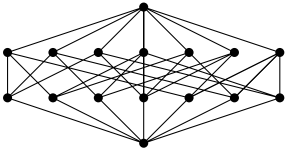
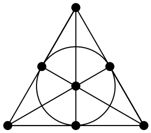
Definition $\mathrm{Sub}(V)^* =$ all elements of height ≤ 2 in Sub(V).

Key idea! $\mathrm{Sub}(V)^*$ is the projective geometry of V.

elements of height $1 =$ points p
elements of height $2 =$ lines ℓ
element of height $0 =$ nothing

The point p lies on the line ℓ iff $p \leq \ell$.

Some examples ...

V	Sub(V)	Sub(V)*
$\mathbb{Z}_2$		
$\mathbb{Z}_2^2$		
$\mathbb{Z}_2^3$		

Projective geometry

Theorem Each of V, Sub(V), Sub(V)* determines the others.[1]

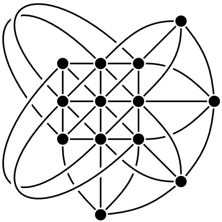
Proof The interesting part, from Sub(V)* to V is via the Greek geometric constructions of arithmetic operations to make the appropriate field.

Note Categorical version (Faure, Frolicher) with morphisms being certain partial maps between projective geometries where kernels and images are subspaces.

[1] Lawyer small print: dim $V > 2$.

Projective geometry

Part of the utility of projective geometry is one knows "locally" how things looks. One does not need, or want, to see all. Even the next smallest example $\text{Sub}(\mathbb{Z}_3^3)^*$ is not so nice!



But one can reason here, or in $\text{Sub}(\mathbb{R}^3)^*$, using geometric tools.

Aim

We seek a similar program, but with ...

- an $\textsc{Omp}$ A
- its poset BSub(A) of Boolean subalgebras
- its elements $\text{BSub}(A)^*$ of height at most 2

Orthomodular posets

Definition An orthoalgebra (OA) is a bounded poset with partially defined operation $\oplus$ that is commutative, associative, and

1. for each x there is a unique x' with $x \oplus x' = 1$
2. $x \oplus x$ is defined iff $x = 0$

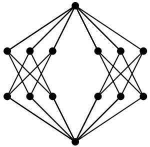
An orthomodular poset (OMP) is an orthoalgebra where $x \oplus y$ is their join when defined.

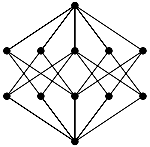
Note Here we keep to OMPs, but our results extend to OAs.

Orthomodular posets

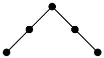
Every OMP poset is built by "gluing" Boolean algebras. It is the geometry of this gluing that comprises the study of OMPs.

Example Two 8-element BAs glued at 0,1 and then at 0,1, an atom and a coatom.





Greechie diagrams (top view of atoms)



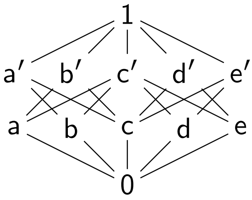
Orthomodular posets

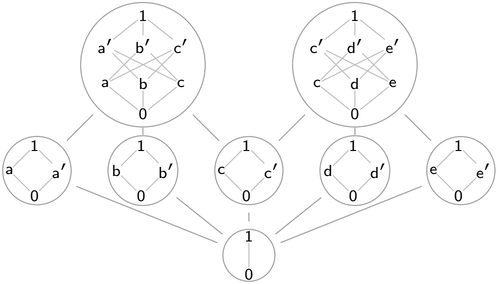
These toy examples are to illustrate ideas. Interesting examples are highly complex.

- The projections of a Hilbert space
- The projections of any von Neumann or C^*-algebra
- The idempotents of any ring
- The direct product decompositions of any set, group, etc.

Boolean subalgebras of an OMP

Definition BSub(A) is the collection of Boolean subalgebras of A partially ordered by set inclusion.





This shows A and BSub(A). While the picture shows the "insides" of each element of BSub(A), we treat BSub(A) simply as a poset.

BSub(A)

The origin of BSub(A) lies in the topos approach to quantum mechanics of Isham et. al. They used not only the abstract order structure of BSub(A), but also knowledge of the actual subalgebras that comprise it.

Later it was shown indirectly that the order structure of BSub(A) determines A. A corresponding result showed that a VN algebra is determined up to its Jordan structure by its poset of abelian subalgebras. The situation for C^* algebras is ongoing.

Physically, elements of BSub(A) are "classical snapshots" of the system.

BSub(A)

Proposition For an OMP A, BSub(A) has the following properties

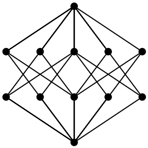
- $\{0, 1\}$ is the least element of BSub(A).
- Atoms of BSub(A) are the sets $\{0, a, a', 1\}$ where $a \neq 0, 1$
- Elements of height n are 2^{n+1}-element Boolean subalgebras
- Atomistic, $\wedge$-semilattice†, with directed joins
- Principle downsets of compact elements are partition lattices.

† This does not hold for orthoalgebras.

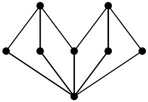
BSub(A)*

Definition BSub(A)* is the poset of Boolean subalgebras of A with at most 8 elements.

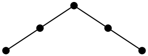
Draw geometrically with points for atoms and lines for elements of height 2. Each line contains exactly 3 points.



A



BSub(A)

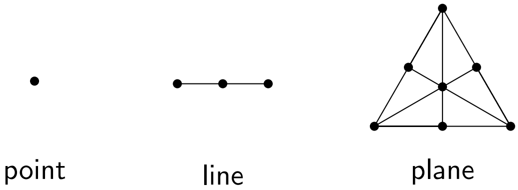


BSub(A)*

In this simple case, BSub(A)* “is” the Greechie diagram!

The Boolean case

Below are the geometric views of $\mathrm{BSub}(A)^*$ where A is a Boolean algebra with 4, 8, 16 elements.

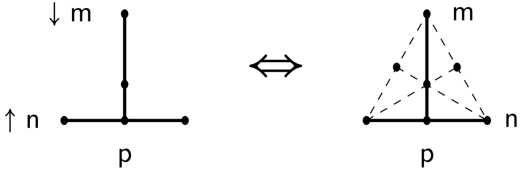


The similarities to the projective geometries of $\mathbb{Z}_2$, $\mathbb{Z}_2^2$ and $\mathbb{Z}_2^3$ are not accidental. Our plane is the Fano plane minus a line.

Proposition The poset of subalgebras of a 2^n element Boolean algebra can be embedded in the lattice of subspaces of $\mathbb{Z}_2^{n-1}$.

Recovering A from BSub(A)*

This relies on the key notion of a "direction" d of a point p. This is an assignment d(m) of either ↑ or ↓ to each line m that contains p.

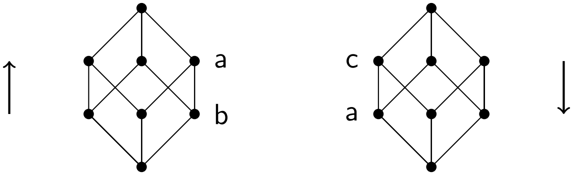


We require that d takes different values of ↑, ↓ on two lines m, n containing p iff m, n are coplanar and are the only lines containing p in the plane they determine.

Idea a 4-element Boolean algebra $\{0, a, a', 1\}$ (point) can sit inside an 8-element one (line) in two ways

- a atom (↓)
- a coatom (↑).

If a is a coatom of one, and an atom of another,



both sit in a 16-element Boolean subalgebra (plane) generated by the chain $0, b, a, c, 1$.

Recovering A from BSub(A)*

Definition For P the geometric diagram of BSub(A)* let Dir(P) be the set of all directions of points of P plus two new elements 0,1.

Definition Put operations $'$ and $\oplus$ on Dir(P) as follows:

1. d' is obtained by switching $\uparrow$ and $\downarrow$ from d.
2. If d is a direction for p and e is a direction for q, then $d \oplus e$ is defined iff p, q lie on a line ℓ and $d(\ell) = \downarrow = e(\ell)$. Then $d \oplus e$ is the direction for the third point r on ℓ with $(d \oplus e)(\ell) = \uparrow$.

Theorem If A is a non-trivial $\mathrm{O{\small MP}}$, then $A \simeq \mathrm{Dir}(P)$.

Characterizing posets BSub(A) and BSub(A)*

We characterize posets of the form BSub(A)* for some OMP A in terms of their geometric description

- Every line has exactly 3 points
- Two distinct points lie on at most one line
- Each triangle lies in a plane
- Every point has a direction

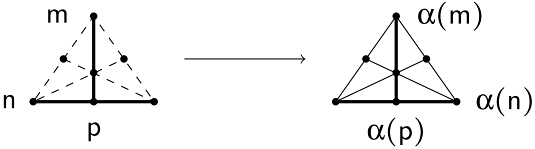
We can also characterize posets of the form BSub(A), but this is more technical. We require that certain joins exist.

A categorical view

Definition An orthohypergraph is a configuration of points and lines that is isomorphic to one arising from some orthomodular poset A.

Definition A partial map α from the points of an orthohypergraph G to those of H is a hypergraph morphism if

- The undefined points form a subspace and the image of a subspace is a subspace.
- If m, n intersect in p and $\alpha(m), \alpha(n)$, span a plane π as shown, then m, n span a plane that is mapped isomorphically onto π.



Near equivalence

For a morphism $f : A \to C$ of orthomodular posets, the image of a Boolean subalgebra of A is a Boolean subalgebra of C.

Theorem There is a functor $\mathcal{G} : \textsc{Omp} \to \mathrm{OH}$ taking an $\textsc{Omp}$ A to the hypergraph associated with $\mathrm{BSub}(A)^*$ and taking morphisms to the corresponding direct image map.

There are small obstacles to providing an equivalence. The 1 and 2-element $\textsc{Omps}$ have the same empty hypergraph, and a 4 element Boolean algebra has 2 automorphisms while its 1-point hypergraph has only one. Modulo these ...

Near equivalence

Theorem The functor $\mathcal{G}\colon \textsc{Omp} \to \mathrm{OH}$ has the following properties:

1. it is surjective on objects
2. it is injective on non-trivial objects
3. it is full and faithful on morphisms where the image of each maximal Boolean subalgebra has more than 4 elements

Comparison with Greechie diagrams

Greechie diagrams are an older technique to represent an OMP using points for atoms and lines to indicate the atoms of a maximal Boolean subalgebra.

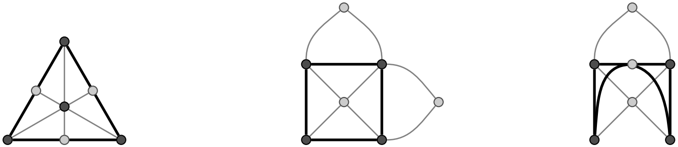
- applies only to chain-finite OMPs
- has fewer points, and lines can be larger than 3 points
- has no mechanism to deal with morphisms
- takes experience to visualize “missing” elements

Lets compare a Greechie diagram with our hypergraph in some less trivial examples. Earlier comments about pictures and projective geometry apply here too!

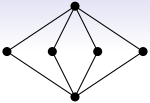
Pictures

In drawing pictures of $\mathrm{BSub}(A)^*$ we can draw lines as curves.

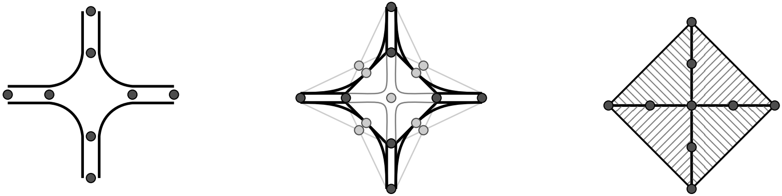
We can also draw the plane in several isomorphic ways.



Consider the OMP MO_2

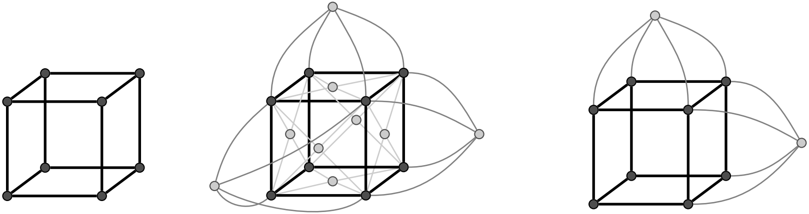


If we consider $MO_2 \times MO_2$ the result has 36 elements, 8 atoms, and two central elements $(1,0)$ and $(0,1)$. Its Greechie diagram, hypergraph, and a modified hypergraph are below.



You don't want all the detail in the hypergraph, but you can reason precisely in it to see the "hidden" feature of most interest, the central elements.

The Fraser cube is an OA with 36 elements and 8 atoms, but with 6 maximal Boolean subalgebras, rather than 4.



We can see from the hypergraph there are Boolean subalgebras whose intersection is not Boolean. The front and back faces are Boolean algebras that share only 2 points, so their intersection cannot be Boolean.

Topos quantum mechanics

Let A be the OMP of projections of a von Neumann algebra $\mathcal{N}$.

The topos of presheaves on BSub(A) is the central ingredient.

Idea Abelian subalgebras of $\mathcal{N}$ correspond to "classical snapshots" of the quantum system. The topos glues these together to encode the quantum version that is built from classical components.

Example The state space of the system is the spectral presheaf Σ that assigns the Stone space Σ_B to each $B \in \mathrm{BSub}(\mathcal{N})$.

Example The Kochen-Specker theorem is equivalent to Σ not having a global section.

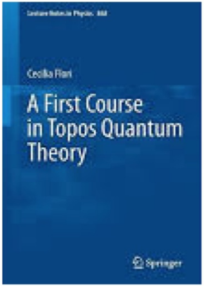
Topos quantum mechanics

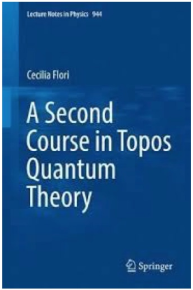
It seems that much (all?) of this approach can be done with the topos of presheaves over $\mathrm{BSub}(\mathcal{N})^*$.

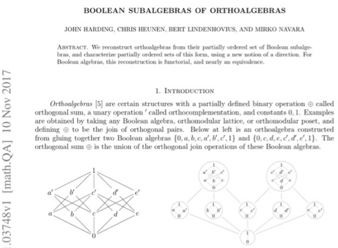
- Some aspects greatly simplify (clopen subobjects of Σ)
- Provides a geometrical view of the base space
- Allows for morphisms between topoi for different systems

Slogan You don't need 10mp classical snapshots, just "glimpses".

Thank you







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BOOLEAN SUBALGEBRAS OF ORTHOALGEBRAS

JOHN HARDING, CHRIS HEUNEN, BERT LINDENHOVIUS, AND MIRKO NAVARA

ABSTRACT. We reconstruct orthoalgebras from their partially ordered set of Boolean subalgebras, and characterize partially ordered sets of this form, using a new notion of a direction. For Boolean algebras, this reconstruction is functorial, and nearly an equivalence.

1. INTRODUCTION

Orthoalgebras [5] are certain structures with a partially defined binary operation $\oplus$ called orthogonal sum, a unary operation $'$ called orthocomplementation, and constants $0, 1$. Examples are obtained by taking any Boolean algebra, orthomodular lattice, or orthomodular poset, and defining $\oplus$ to be the join of orthogonal pairs. Below at left is an orthoalgebra constructed from gluing together two Boolean algebras $\{0, a, b, c, a', b', c', 1\}$ and $\{0, c, d, e, c', d', e', 1\}$. The orthogonal sum $\oplus$ is the union of the orthogonal join operations of these Boolean algebras.