The convolution algebra

John Harding, Carol and Elbert Walker

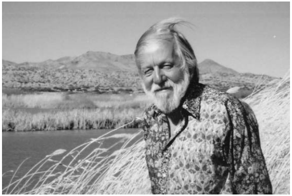
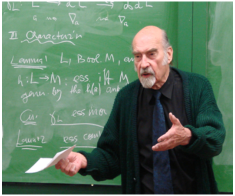
New Mexico State University
www.math.nmsu.edu/~JohnHarding.html

jharding@nmsu.edu

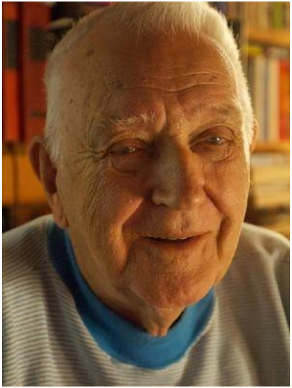
Prague, June 2017

Dedication

I've been fortunate to have several wonderful people to give me advice through the years.


Bjarni Jónsson



I did a postdoc under Bjarni, 1991-93 in Nashville

Keith Kearnes and Peter Jipsen were there, Mai Gehrke had just left

Introduction

I learned about complex algebras, canonical extensions, and BAOs under Bjarni.

Of course, it was Jónsson and Tarski that did the pathbreaking work on the subject, originally motivated by relation algebras.

I'd like to present a small observation here. It may be known, but wasn't to me and several I asked. Like many things in the subject, the proofs mostly write themselves once the ideas are in place.

Complex algebras

For a relational structure $\mathfrak{X} = (X, (R_i)_I)$ where R_i is $(n_i + 1)$-ary, its complex algebra $\mathfrak{X}^+$ is the power set $\mathcal{P}(X)$ with the n_i-ary operations f_i where

$$f_i(A_i, \ldots, A_{n_i}) = \{x : \exists (x_1, \ldots, x_{n_i}, x) \in R_i \text{ with } x_j \in A_j \text{ each } j\}$$

Its dual complex algebra $\mathfrak{X}^-$ is $(\mathcal{P}(X), (g_i)_I)$ where

$$g_i(A_1, \ldots, A_{n_i}) = \{x : \forall (x_1, \ldots, x_{n_i}, x) \in R_i \Rightarrow \exists j \text{ with } x_j \in A_j\}$$

Its double complex algebra $\mathfrak{X}^*$ is $(\mathcal{P}(X), (f_i)_I, (g_i)_I)$.

Complex algebras

In the terminology of Jónsson and Tarski, the complex algebra $\mathfrak{X}^+$ is a Boolean algebra with operators $\textsc{Bao}$. The operations g_i are dual to the f_i in that

$$g_i(A_1, \ldots, A_{n_i}) = \neg f_i(\neg A_1, \ldots, \neg A_{n_i})$$

Here $\neg A = X \setminus A$ is Boolean negation.

Example

If you consider a group $\mathcal{G} = (G, \cdot, ^{-1}, e)$ as a relational structure with a ternary, binary, and unary relation, its complex algebra is the usual group complex $\mathcal{G}^+$ where

$$
\begin{aligned}
A\,;B &= \{ab : a \in A, b \in B\} \\
A^{\smile} &= \{a^{-1} : a \in A\} \\
1' &= \{e\}
\end{aligned}
$$

These have been studied apparently since Frobenius. They are basic examples of relation algebras.

Basic Definitions

Definition A type τ is a mapping $\tau : \mathrm{I} \to \mathbb{N}$ for some set I

Definition A relational structure $\mathfrak{X} = (X, (R_i)_I)$ of type τ is a set X with a family of relations on X where R_i is $\tau(i) + 1$-ary.

Definition A lattice algebra of type τ is a complete lattice with a family $(f_i)_I$ of additional operations where f_i is $\tau(i)$-ary.

Note For a relational structure $\mathfrak{X}$ of type τ, its complex algebra $\mathfrak{X}^+$ and dual complex algebra $\mathfrak{X}^-$ are lattice algebras of type τ.

The Convolution Algebra

Definition For a relational structure $\mathfrak{X}$ of type τ and a complete lattice L, the convolution of $\mathfrak{X}$ by L is the lattice algebra $L^{\mathfrak{X}}$ whose underlying lattice is the complete lattice L^X where the additional operation f_i is given by

$$f_i(\alpha_1, \ldots, \alpha_n)(x) = \bigvee\{\alpha_1(x_1) \wedge \cdots \wedge \alpha_{n_i}(x_{n_i}) : (x_1, \ldots, x_{n_i}, x) \in R_i\}$$

The dual convolution algebra $L^{\mathfrak{X}-}$ has additional operations

$$g_i(\alpha_1, \ldots, \alpha_n)(x) = \bigwedge\{\alpha_1(x_1) \vee \cdots \vee \alpha_{n_i}(x_{n_i}) : (x_1, \ldots, x_{n_i}, x) \in R_i\}$$

The double convolution algebra has both $L^{\mathfrak{X}*} = (L^X, (f_i)_I, (g_i)_I)$.

Basic Observation

Proposition Let $\mathfrak{X}$ be a relational structure of type τ and 2 be the two-element lattice.

1. $\mathfrak{X}^{+} \simeq 2^{\mathfrak{X}}$
2. $\mathfrak{X}^{-} \simeq 2^{\mathfrak{X}-}$
3. $\mathfrak{X}^{*} \simeq 2^{\mathfrak{X}*}$

First Properties

Definition An n-ary operation f on a lattice L is an operator if for each coordinate $i \leq n$ and each finite subset $S \subseteq L$

$$f(a_1, \ldots, \bigvee S, \ldots, a_n) = \bigvee \{f(a_1, \ldots, s, \ldots, a_n) : s \in S\}$$

Definitions of complete operator, dual operator and complete dual operator are obvious.

Proposition Let L be a complete distributive lattice and $\mathfrak{X}$ be a relational structure. Then the operations of $L^{\mathfrak{X}}$ are operators. If L is is the lattice reduct of a complete Heyting algebra, they are complete operators. Similarly for $L^{\mathfrak{X}-}$ and dual operators.

First Properties

Complex algebras $\mathfrak{X}^+$ are atomic, and since their operations are complete operators, they are determined by their action on atoms.

If L is atomless, so also will be the convolution $L^{\mathfrak{X}}$. But we do have the following.

Proposition In a convolution algebra $L^{\mathfrak{X}}$, the value of an operation f is determined by its action on elements $\alpha \in L^X$ of finite support.

Categorical Aspects

Definition Lat is the category whose objects are complete lattices and whose morphisms are maps that preserve $\wedge, \bigvee$. Let Lat^{-} have the same objects with maps that preserve $\vee, \bigwedge$ and Lat^{*} have the same objects with maps that preserve $\bigwedge, \bigvee$.

Definition Rel_τ is the category of relational structures of type τ with morphisms being *p*-morphisms.

Definition Alg_τ is the category whose objects are complete lattices with additional operations of type τ and homomorphisms preserving $\wedge, \vee$ and the additional operations.

Categorical Aspects

Theorem There is a bifunctor, covariant in the first argument and contravariant in the second

$$\mathsf{Conv}(\,\cdot\,,\,\cdot\,) : \mathsf{Lat} \times \mathsf{Rel}_\tau \to \mathsf{Alg}_\tau$$

For objects it takes $L, \mathfrak{X}$ to $L^{\mathfrak{X}}$
For morphisms it takes $f : L \to M$, $p : \mathcal{Y} \to \mathfrak{X}$ to $\phi : L^{\mathfrak{X}} \to M^{\mathcal{Y}}$

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "p"] \arrow[d, "\alpha"'] & \mathcal{Y} \arrow[d, "\phi(\alpha)"] \\
L \arrow[r, "f"] & M
\end{tikzcd}$$

This functor Conv

1. preserves products in the 1^{st} argument
2. takes coproducts to products in the 2^{nd} argument
3. preserves and reflects one-one and onto maps in 1^{st} argument
4. takes one-one to onto and onto to one-one in 2^{nd} argument

These results have obvious modification to

$$\mathrm{Conv}^{-} : \mathrm{Lat}^{-} \times \mathrm{Rel}_{\tau} \to \mathrm{Alg}_{\tau}$$

$$\mathrm{Conv}^{*} : \mathrm{Lat}^{*} \times \mathrm{Rel}_{\tau} \to \mathrm{Alg}_{\tau}$$

There are also versions where $\mathfrak{X} = (X, \leq, (R_i)_I)$ is an ordered relational structure and $L^{\mathfrak{X}}$ consists of order-preserving maps.

Preservation of identities

Correspondence theory links properties of frames $\mathfrak{X}$ to equational properties of their complex algebras $\mathfrak{X}^+$. We seek a similar thing for convolution algebras.

Theorem If L is non-trivial and completely distributive, then $L^{\mathfrak{X}_*}$ and $\mathfrak{X}^*$ satisfy the same equations in $\wedge, \vee, 0, 1, (f_i)_I, (g_i)_I$.

Proof Working in Lat^* and using a result of Raney characterizing completely distributive lattices,

$$
\begin{aligned}
2 \in S(L) \quad &\Rightarrow \quad 2^{\mathfrak{X}_*} \in S(L^{\mathfrak{X}_*}) \\
L \in HSP(2) \quad &\Rightarrow \quad L^{\mathfrak{X}_*} \in HSP(2^{\mathfrak{X}_*})
\end{aligned}
$$

The result follows from $\mathfrak{X}^* \simeq 2^{\mathfrak{X}_*}$.

Preservation of identities

Theorem If L is a non-trivial spatial frame, then $L^{\mathfrak{X}}$ and $\mathfrak{X}^{+}$ satisfy the same equations in $\wedge, \vee, 0, 1, (f_i)_I$.

Proof In Lat spatial frames are those in $SP(2)$.

More is true, but the proof of the following requires properties of operators and finitely supported elements.

Theorem If L is a non-trivial frame, then $L^{\mathfrak{X}}$ and $\mathfrak{X}^{+}$ satisfy the same equations in $\wedge, \vee, 0, 1, (f_i)_I$.

Theorem If L is the dual of a non-trivial frame, then $L^{\mathfrak{X}-}$ and $\mathfrak{X}^{-}$ satisfy the same equations in $\wedge, \vee, 0, 1, (g_i)_I$.

Negative results on equations

While $L^{\mathfrak{X}}$ is defined for any lattice, it behaves poorly with respect to equations without distributivity.

Proposition For $\mathfrak{X} = (\mathbb{Z}_2, +)$ these are equivalent.

1. the operation $+$ on $L^{\mathfrak{X}}$ is associative
2. L is distributive

Proposition Let ∇_X be the universal relation on a set X. These are equivalent.

1. $L^{(X,\nabla_X)}$ satisfies $f(a) \wedge f(b) = f(f(a) \wedge b)$ for each each set X
2. L is a frame

Examples

Example 1 (1953) Foster's original definition of a bounded Boolean power is similar to $L^{\mathfrak{X}}$ when L is Boolean and $\mathfrak{X}$ is an algebra. However, he considers only certain functions $\alpha : X \to L$.

Example 2 (1957) Monteiro and Varsavsky's functional monadic Heyting algebras are $L^{(X,\nabla_X)}$ when L is a Heyting algebra.

Note, G. Bezhanishvili and I showed every monadic Heyting algebra is a subalgebra of a functional one.

Example 3 (1975) Zadeh defined type-2 fuzzy sets to be $\mathsf{I}^{\mathfrak{I}}$ where I is the unit interval and $\mathfrak{I} = (\mathsf{I}, \wedge, \vee, \neg, 0, 1)$.

Note, since I is completely distributive, $\mathsf{I}^{\mathfrak{I}}$ behaves well with respect to preservation of equations.

Back to the relation algebras that motivated Jónsson and Tarski.

Theorem For a group $\mathfrak{G} = (G, \cdot, ^{-1}, e)$ and frame L, $L^{\mathfrak{G}}$ satisfies

1. $a; (b; c) = (a; b); c$
2. $a; 1' = a = 1'; a$
3. $(a \vee b); c = (a; c) \vee (b; c)$ and $a; (b \vee c) = (a; b) \vee (a; c)$
4. $(a^{\smile})^{\smile} = a$
5. $(a \vee b)^{\smile} = a^{\smile} \vee b^{\smile}$
6. $(a; b)^{\smile} = b^{\smile}; a^{\smile}$
7. $(a^{\smile}; \neg(a; b)) \vee \neg b = \neg b$
8. $a; b \leq \neg\neg c \Leftrightarrow a^{\smile}; \neg c \leq \neg b \Leftrightarrow \neg c; b^{\smile} \leq \neg a$

These are the usual axioms for a relation algebra except we have a Heyting algebra, and in (8) the $\neg\neg c$ is usually just c. It satisfies (8) with $\neg\neg c$ replaced by c iff L is a Boolean algebra.

Problems

Problem 1 If L is both a frame and a dual frame, do $L^{\mathfrak{X}*}$ and $\mathfrak{X}^*$ satisfy the same equations in $\wedge, \vee, 0, 1, (f_i)_I, (g_i)_I$? In particular, is this true for L a complete Boolean algebra?

Problem 2 Include the componentwise Heyting negation $\neg$ in considerations of equations when L is a frame.

Problem 3 I suspect there is a connection to topoi. Develop this.

Thank you Bjarni

A draft of this paper is on the ArXiv

Other papers at www.math.nmsu.edu/~jharding