Some Quantum Logic and a few Categories

John Harding

New Mexico State University
www.math.nmsu.edu/JohnHarding.html

jharding@nmsu.edu

Workshop on Categorical Quantum Logic
Oxford, August 2007

Organization

Part 1	Background on Quantum Logic
Part 2	My view of things

Part 1: What is Quantum Logic?

To me, quantum logic is the use of well-motivated mathematical structures to study foundational aspects of quantum mechanics.

We focus on one path ... the connection between questions of a quantum system and orthomodular posets.

Part 1: What are Questions?

Definition Questions of a quantum system are measurements that have two distinguished outcomes, usually called *Yes* and *No*.

Examples

- Is spin up?
- Is position in the right half-plane?

One feels there is a sort of logic for the questions. For instance, you may have an idea what the negation of a question might be.

Part 1: Orthomodular Posets

Definition $(P, \leq, \perp)$ is an orthomodular poset (OMP) if

1. $(P, \leq)$ is a bounded poset
2. $\perp$ is an order-inverting, period two, complementation
3. If $x \leq y^\perp$ then x, y have a join
4. If $x \leq y^\perp$ then $x \vee (x \vee y)^\perp = y^\perp$ (the orthomodular law)

An OMP that is a lattice is an orthomodular lattice (OML)

Part 1: Orthomodular Posets

Definition $(P, \leq, \perp)$ is an orthomodular poset (OMP) if

1. $(P, \leq)$ is a bounded poset
2. $\perp$ is an order-inverting, period two, complementation
3. If $x \leq y^{\perp}$ then x, y have a join
4. If $x \leq y^{\perp}$ then $x \vee (x \vee y)^{\perp} = y^{\perp}$ (the orthomodular law)

An OMP that is a lattice is an orthomodular lattice (OML)

Short story An OMP is a bunch of Boolean algebra glued together.

Part 1: Brief History

Part 1: Brief History

1932 von Neumann formulates Q.M. in terms of Hilbert space

$$\text{Borel } \mathbb{R} \xrightarrow{\Sigma} \text{Proj } \mathbb{H} \xrightarrow{\mu} [0, 1]$$

Part 1: Brief History

1932 von Neumann formulates Q.M. in terms of Hilbert space

$$\text{Borel } \mathbb{R} \xrightarrow{\Sigma} \text{Proj } \mathbb{H} \xrightarrow{\mu} [0,1]$$

1936 Birkhoff, von Neumann: questions are a subalg of Proj $\mathbb{H}$. Other models for the questions should be considered.

Part 1: Brief History

1932 von Neumann formulates Q.M. in terms of Hilbert space

$$\text{Borel } \mathbb{R} \xrightarrow{\Sigma} \text{Proj } \mathbb{H} \xrightarrow{\mu} [0,1]$$

1936 Birkhoff, von Neumann: questions are a subalg of Proj $\mathbb{H}$. Other models for the questions should be considered.

1950-70 A rich theory of OMLs was developed.

1965 Plausible axioms to show questions form OMP.

1970's Pathological examples, no tensor product for OMPs.

1980- Ever more general structures.

Part 1: Brief History

1932 von Neumann formulates Q.M. in terms of Hilbert space

$$\text{Borel } \mathbb{R} \xrightarrow{\Sigma} \text{Proj } \mathbb{H} \xrightarrow{\mu} [0,1]$$

1936 Birkhoff, von Neumann: questions are a subalg of Proj $\mathbb{H}$. Other models for the questions should be considered.

1950-70 A rich theory of OMLs was developed.

1965 Plausible axioms to show questions form OMP.

1970's Pathological examples, no tensor product for OMPs.

1980- Ever more general structures.

2002 Bob, Samson: categorical approach for compound systems.

Bob and Samson's approach

Setting **C** is strongly compact closed category with biproducts.

They develop much of Q.M. in this setting, with a novel treatment of interacting systems. This differs greatly from the quantum logic stream, still ...

Bob and Samson's approach

Setting **C** is strongly compact closed category with biproducts.

They develop much of Q.M. in this setting, with a novel treatment of interacting systems. This differs greatly from the quantum logic stream, still ...

Proposition For $A \in \mathbf{C}$, the questions of A lie in an OMP.

Bob and Samson's approach

Setting **C** is strongly compact closed category with biproducts.

They develop much of Q.M. in this setting, with a novel treatment of interacting systems. This differs greatly from the quantum logic stream, still ...

Proposition For $A \in \mathbf{C}$, the questions of A lie in an OMP.

Examples

- In the category **Rel** this OMP is the power set of A.
- In the category **FdHilb** this OMP is Proj A.

Part 1: Biased Summary

Slogan Modularity has a rich mathematical theory because it captures a primitive notion — projective geometry.

Slogan Orthomodularity has a rich mathematical theory.

Slogan Orthomodularity is tied to questions of a Q.M. system.

Part 2

My view of things

Part 2: Objectives

1. Show orthomodularity has a primitive mathematical root.
2. Use this root to build physical axioms for the questions.
3. Possibilities for using this in categorical approaches.

Part 2: Binary Decompositions

Setting S is a set (or group, or top space, or Hilbert space, ...)

Definition A binary product map is an iso $f : S \to S_1 \times S_2$.

Definition Two such maps are equivalent if there are iso's i, j

$$\begin{array}{ccc} & \xrightarrow{f} & S_1 \times S_2 \\ S & & \downarrow i \quad \downarrow j \\ & \xrightarrow[g]{} & T_1 \times T_2 \end{array}$$

Definition A binary decomposition of S is an equivalence class

$$[S \cong_f S_1 \times S_2].$$

Part 2: Decompositions and Orthomodularity

Definition Let BDec S be all binary decompositions of S and let

1. $0 = [S \cong \{*\} \times S]$
2. $1 = [S \cong S \times \{*\}]$

Part 2: Decompositions and Orthomodularity

Definition Let BDec S be all binary decompositions of S and let

1. $0 = [S \cong \{*\} \times S]$
2. $1 = [S \cong S \times \{*\}]$
3. $\perp$ is the operation $[S \cong S_1 \times S_2]^{\perp} = [S \cong S_2 \times S_1]$

Part 2: Decompositions and Orthomodularity

Definition Let BDec S be all binary decompositions of S and let

1. $0 = [S \cong \{*\} \times S]$
2. $1 = [S \cong S \times \{*\}]$
3. $\perp$ is the operation $[S \cong S_1 \times S_2]^\perp = [S \cong S_2 \times S_1]$
4. $\leq$ is the relation $[S \cong S_1 \times (S_2 \times S_3)] \leq [S \cong (S_1 \times S_2) \times S_3]$

Part 2: Decompositions and Orthomodularity

Definition Let BDec S be all binary decompositions of S and let

1. $0 = [S \cong \{*\} \times S]$
2. $1 = [S \cong S \times \{*\}]$
3. $\perp$ is the operation $[S \cong S_1 \times S_2]^\perp = [S \cong S_2 \times S_1]$
4. $\leq$ is the relation $[S \cong S_1 \times (S_2 \times S_3)] \leq [S \cong (S_1 \times S_2) \times S_3]$

Theorem BDec S is an OMP.

Part 2: Decompositions and Orthomodularity

Definition Let BDec S be all binary decompositions of S and let

1. $0 = [S \cong \{*\} \times S]$
2. $1 = [S \cong S \times \{*\}]$
3. $\perp$ is the operation $[S \cong S_1 \times S_2]^\perp = [S \cong S_2 \times S_1]$
4. $\leq$ is the relation $[S \cong S_1 \times (S_2 \times S_3)] \leq [S \cong (S_1 \times S_2) \times S_3]$

Theorem BDec S is an OMP.

The source of orthomodularity direct product decompositions

Part 2: Examples of BDec

Some standard methods of constructing OMPs are special cases

- Proj $\mathbb{H}$ for a Hilbert space
- the splitting subspaces of an inner product space
- the idempotents of a ring R
- OMPs from modular lattices

Part 2: Physical setting

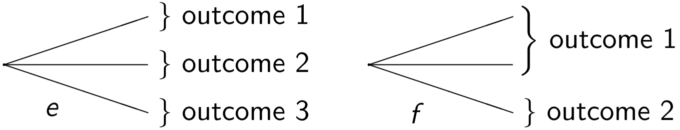
Setup E is the set of experiments of a physical system. Assume

- each experiment $e \in E$ has finitely many outcomes
- the outcomes are mutually exclusive and exhaustive
- the outcomes are called outcome 1, ..., outcome n.

Part 2: New Experiments from Old

Example If e is ternary, we can build a binary f as follows:

- combine outcomes 1,2 of e and call this outcome 1 of f.
- let outcome 3 of e be outcome 2 of f.



Write $f = (\{1,2\},\{3\})e$.

Call $(\{1,2\},\{3\})$ a partition of 3, $(\{2\},\{3\},\{1\})$ is another.

Part 2: Decompositions and Probabilities

Definition Dec S is all decompositions $[S \cong S_1 \times \cdots \times S_n]$.

Definition Prob S is all maps $p : S \to [0,1]^n$ where for each $s \in S$

$$p_1(s) + \cdots + p_n(s) = 1$$

Again, we can build new from old.

Example

- $(\{1,2\},\{3\})[S \cong S_1 \times S_2 \times S_3] = [S \cong (S_1 \times S_2) \times S_3]$
- $(\{1,2\},\{3\})(p_1, p_2, p_3) = (p_1 + p_2, p_3)$

Part 2: Axioms of an Experimental System

Definition An experimental system is a map $D : E \longrightarrow \text{Dec } S$ where

1. If e is an n-ary experiment, De is an n-ary decomposition
2. $D(\sigma e) = \sigma(De)$ when defined.

Part 2: Axioms of an Experimental System

Definition An experimental system is a map $D : E \longrightarrow \text{Dec } S$ where

1. If e is an n-ary experiment, De is an n-ary decomposition
2. $D(\sigma e) = \sigma(De)$ when defined.

It has probabilities if there is a map $P : E \longrightarrow \text{Prob } S$ with

4. If e is an n-ary experiment, Pe is an n-ary probability map.
5. $P(\sigma e) = \sigma(Pe)$ when defined.

Part 2: Axioms of an Experimental System

Definition An experimental system is a map $D : E \longrightarrow \text{Dec } S$ where

1. If e is an n-ary experiment, De is an n-ary decomposition
2. $D(\sigma e) = \sigma(De)$ when defined.

It has probabilities if there is a map $P : E \longrightarrow \text{Prob } S$ with

4. If e is an n-ary experiment, Pe is an n-ary probability map.
5. $P(\sigma e) = \sigma(Pe)$ when defined.

A few bits left out, such as $(\{1\}, \{2\})e = e$ etc.

Part 2: The Standard Model

For $\mathbb{H}$ a Hilbert space,

- An n-ary experiment e gives closed subspaces $A_1, \ldots, A_n$
- De is the decomposition $\mathbb{H} \cong A_1 \times \cdots \times A_n$
- Pe is the n-ary probability map $(p_1, \ldots, p_n)$ where

$$p_i(s) = \frac{\|s_{A_i}\|^2}{\|s\|^2}$$

  here s_{A_i} is the projection of s onto A_i.

This gives an experimental system with probabilities.

Part 2: Questions of an Experimental System

Theorem The set Ques E of binary experiments forms an OMP

Definition A set of questions is compatible if they can be built from a common experiment f.

Theorem For a finite subset $A \subseteq$ Ques E these are equivalent

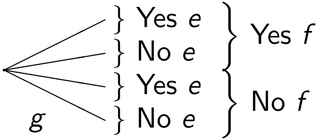
1. A is compatible
2. Any two elements of A are compatible
3. A is contained in a Boolean subalgebra of Ques E

There is some content to this, physical and mathematical.

Part 2: The Logic of Questions

Proposition For e, f compatible there is a unique g with

$$e = (\{1,3\},\{2,4\})g$$
$$f = (\{1,2\},\{3,4\})g$$



Definition For e, f compatible, set $e \text{ OR } f = (\{1,2,3\},\{4\})g$ etc.

Proposition When defined, we get such familiar rules as

1. $e \text{ OR } (f \text{ AND } g) = (e \text{ OR } f) \text{ AND } (e \text{ OR } g)$
2. $\text{NOT } (e \text{ OR } f) = (\text{NOT } e) \text{ AND } (\text{NOT } f)$.
3. etc.

Part 2: Observables

Cavemen know position means is it here, or is it here, or is it here

- Position is a word for a family of compatible questions.
- Position in an interval can be measured. Position at a point is an ideal concept for a maximally consistent set of questions.
- Assigning numbers to “ideal questions” is called a scaling.

Part 2: Observables

Cavemen know position means is it here, or is it here, or is it here

- Position is a word for a family of compatible questions.
- Position in an interval can be measured. Position at a point is an ideal concept for a maximally consistent set of questions.
- Assigning numbers to "ideal questions" is called a scaling.

Definition

1. An observable quantity is a Boolean subalg B of Ques E.
2. Ideal questions are points of the Stone space X of B.
3. A scaling is a measurable map $f : X \to \mathbb{R} \cup \{\pm\infty\}$.

Part 2: Computing with observables

Proposition Each state $s \in S$ gives a probability measure μ_s on X.

Definition For an observable quantity B with scaling f

$$\begin{aligned} \mu_s(f^{-1}(U)) &= \text{probability of a result in } U \text{ when in state } s \\ \int_X f d\mu_s &= \text{the expected value} \end{aligned}$$

Hilbert spaces: Self-adjoint A give B, X, f. (Spectral theorem)

Part 2: Biased Opinions

- The axioms for an experimental system have only one fancy assumption, and this uses a primitive idea — decompositions
- From these axioms we get many features of isolated systems
- This addresses only part of the problem — what about interactions?

Part 2: Use in Categorical Quantum Logic?

Products are a purely categorical notion. But assumptions on the category are needed for the algebra of decomp's to behave well.

Definition A category **C** is honest if it has finite products and

$$\begin{array}{ccc} & A \times B \times B & \\ \swarrow & & \searrow \\ A \times B & & B \times C \\ \searrow & & \swarrow \\ & B & \end{array} \qquad \text{is a pushout}$$

Theorem For **C** honest, Dec A forms an orthoalgebra.

Part 2: Why Decompositions?

Dirac explained why vector spaces in his monograph

> *The superposition process is a kind of additive process and implies that states can in some way be added to give new states. The states must therefore be connected with mathematical quantities of a kind which can be added together to give other quantities of the same kind. The most obvious of such quantities are vectors.*

Part 2: Why Decompositions?

Dirac explained why vector spaces in his monograph

> *The superposition process is a kind of additive process and implies that states can in some way be added to give new states. The states must therefore be connected with mathematical quantities of a kind which can be added together to give other quantities of the same kind. The most obvious of such quantities are vectors.*

Perhaps Instead of adding $u + v$, form the ordered pair (u, v)

Part 2: Why Decompositions?

Dirac explained why vector spaces in his monograph

> *The superposition process is a kind of additive process and implies that states can in some way be added to give new states. The states must therefore be connected with mathematical quantities of a kind which can be added together to give other quantities of the same kind. The most obvious of such quantities are vectors.*

Perhaps Instead of adding $u + v$, form the ordered pair (u, v)

Decompositions!

Thank you for listening.

Papers at www.math.nmsu.edu/JohnHarding.html

Part 2: Open Problems

- Which OMPs arise as BDec S?
- Explore BDec S for special S (normed groups, etc).
- Develop BDec S in a categorical setting.
- Explore BDec $(S_1 \times S_2)$ and BDec $(S_1 \otimes S_2)$ etc.
- Find conditions on S for BDec S to be well behaved.