# Canonical Completions

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# The Start

AT

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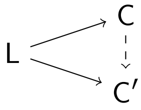
# MacNeille completions

Defn A MacNeille completion of a lattice L is an embedding $L \leq C$ into a complete lattice C that is

- meet dense: $c = \bigwedge\{a \in L : c \leq a\}$
- join dense: $c = \bigvee\{a \in L : a \leq c\}$

Thm Each lattice has a MacNeille completion and it is unique up to unique commuting isomorphism.



Proof. Uniqueness is not hard (try it!). Essentially, each $c \in C$ is determined by the $a \in L$ above and below it.

Existence is usually established by taking normal ideals of L. These are subsets $N \subseteq L$ such that $N = LU(N)$ where

$$U(S) = \{a : s \leq a \text{ for each } s \in S\}$$
$$L(S) = \{a : a \leq s \text{ for each } s \in S\}$$

In other language, normal ideals are the fixed points of the polarity given by the relation $\leq$ on L.

# MacNeille Completions

MacNeille completions have good and bad. They have nice order-theoretic properties, but poor algebraic ones.

- They preserve all existing joins and meets
- The MC of a Boolean algebra is Boolean
- The MC of a Heyting algebra is a Heyting algebra
- Only two varieties of lattices are closed under MC's
- Only three varieties of Heyting algebras are closed under MC's

Quick question: What are the two/three?

# MacNeille Completions and Stone Duality

Thm For a Boolean algebra B with Stone space X its MacNeille completion is given by

$$\begin{array}{ccc} B & \leq & MC(B) \\ \shortparallel & & \shortparallel \\ \text{Clopen}(X) & \leq & \text{Reg}(X) \end{array}$$

Here, Reg(X) is the collection of regular open sets, ones that equal the interior of their closure.

# Canonical Completions

What about the following completion?

$$\begin{array}{ccc} B & \leq & B^\sigma \\ \shortparallel & & \shortparallel \\ \mathrm{Clopen}(X) & \leq & \mathrm{Pow}(X) \end{array}$$

This is called the canonical completion (CC) of B. It is known since Stone. First used by Jónsson and Tarski in 1950's to study relation algebras. Later used in modal logic and Kripke completeness.

# Canonical Completions

Defn For a completion $L \leq C$ we say

- $k \in C$ is closed if it is a meet of elements of $L$
- $u \in C$ is open if it is a join of elements of $L$

Let $K$ be the closed elements and $O$ be the open ones.

Note In the completion $\text{Clopen}(X) \leq \text{Pow}(X)$,

$$\text{Closed} = \text{topologically closed}$$

$$\text{Open} = \text{topologically open}$$

# Canonical Completions

Defn A completion $L \leq C$ of a bounded lattice L is a canonical completion if for each $c \in C$ and $A, B \subseteq L$

1. each $c \in C$ is a join of closed elements
2. each $c \in C$ is a meet of open elements
3. $\bigwedge A \leq \bigvee B \Rightarrow \exists$ finite $A' \subseteq A, B' \subseteq B$ with $\bigwedge A' \leq \bigvee B'$

$1 + 2$ are called density. 3 is called compactness.

Thm Each bounded lattice L has a CC $L^\sigma$ and this is unique up to unique commuting isomorphism.

# Canonical Completions — Boolean Algebras

Prop $\mathsf{Clopen}(X) \leq \mathsf{Pow}(X)$ is a CC.

Proof

Hausdorff $\Rightarrow$ each singleton $\{x\}$ is closed. This gives density.

Topological compactness of $X \Rightarrow$ compactness.

# Canonical Completions — General Case

## Proof of Uniqueness

Compactness shows the poset $K \cup O$ is uniquely determined. Density shows that a canonical completion is $MC(K \cup O)$.

## Proof of Existence

Define sets $\mathcal{F}$, $\mathcal{I}$ and a relation $R \subseteq \mathcal{F} \times \mathcal{I}$ as follows.

$$\mathcal{F} = \{F : F \text{ is a filter of } L\}$$

$$\mathcal{I} = \{I : I \text{ is an ideal of } L\}$$

$$R = \{(F, I) : F \cap I \neq \varnothing\}$$

Then $L^\sigma$ is the fixed points of the polarity given by R.

# Notes on the General Case

Note The existence proof is a template for completions by choosing particular sets of filters and ideals.

- MacNeille completion: $\mathcal{F}, \mathcal{I}$ are principle filters and ideals
- Ideal completion: $\mathcal{F}$ is principle filters, $\mathcal{I}$ is all ideals

Note None of this uses the axiom of choice except the particular realization of $B^{\sigma}$ via Stone duality.

# Extending Maps

Consider an order preserving map $f : L \to M$ between lattices. Using the order dual $L^d$ and finite products, this is quite general.

Example Heyting implication $\to : L^d \times L \to L$ is order preserving.

Defn A lattice expansion (LE) is a bounded lattice with a family of operations that preserve or reverse order in each coordinate.

Many of the strongest results require a map $f : L^n \to L$ to preserve finite joins in each coordinate. These are called operators. They are prominent in modal logic.

# Extending Maps

Defn For $f : L \to M$, let $f^\sigma : L^\sigma \to M^\sigma$ be given as follows

$$f^\sigma(k) = \bigwedge\{f(a) : k \leq a\} \qquad (k \text{ closed})$$
$$f^\sigma(c) = \bigvee\{f^\sigma(k) : k \leq c\}$$

There is a dual notion $f^\pi : L^\sigma \to M^\sigma$ using meets of open elements.

A similar thing can be done for MacNeille completions. Easier.

Defn The MacNeille and canonical completion of a LE $(L, f_i)$ is the completion of the lattice with the extension of its operations.

# Kripke frames

Example Let X be a set with an n+1-ary relation R. Then $f_R$ is an n-ary operator on Pow(X) where

$$f_R(\vec{A}) = \{x \in X : \vec{a}\, R\, x \text{ for some } \vec{a} \in \vec{A}\}$$

# Kripke frames

Thm For $(B, f_i)$ a Boolean algebra with $f_i$ an $n_i$-ary operator, there are $n_i + 1$-ary relations $R_i$ on the Stone space X with

$$(B, f_i)^\sigma \simeq (\mathrm{Pow}(X), f_{R_i})$$

Note  This has as a consequence that a variety of modal algebras that is closed under canonical completions is Kripke complete.

# A Laundry List of Theorems

Thm 1

The classes of lattices, distributive lattices, Boolean algebras, and Heyting algebras are all closed under CCs.

Thm 2

The canonical completion for LEs is functorial.

Thm 3

If a class $\mathcal{K}$ of LEs is closed under CCs and ultraproducts, then the variety $\mathsf{V}(\mathcal{K})$ it generates is closed under CCs.

# A Laundry List of Theorems

Thm 4

Sahlqvist's Theorem provides sufficient syntactic conditions for an equation to be preserved by CCs.

Thm 5

For $(L, f_i)$ a LE, the CC $(L, f_i)^\sigma$ can be embedded into the MC of an ultrapower of $(L, f_i)$.

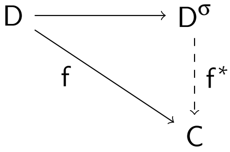
Cor

If a variety of LEs is closed under MCs, then it is closed under CCs.

# A Laundry List of Theorems

### Thm 6

If D is distributive and C is completely distributive, any lattice homo $f : D \to C$ extends to a complete homo $f : D^\sigma \to C$.




### Note

This says that CCs for distributive lattices are free completely distributive extensions. This provides a simple path to various classical results about completely distributive lattices.

# A Laundry List of Theorems

Thm 7

If V is a finitely generated variety of LEs, then the canonical completion is the profinite completion for members of V.

# Thank You!