Decompositions in Quantum Mechanics

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Washington, February 2019

The role of projection operators

In the standard Hilbert space formulation of QM, projections play a central role. Our key ingredients.

$$\mathcal{Q} = \text{the orthomodular lattice of projections of } \mathcal{H}$$
$$\mathcal{S} = \text{the convex set of states}$$
$$\mathcal{O} = \text{the observables}$$
$$\mathcal{B} = \text{the Borel algebra of } \mathbb{R}$$
$$\mathcal{G} = \text{a Lie group}$$

The role of projection operators

The Spectral Theorem

Observables correspond to σ-homomorphisms $A : \mathcal{B} \to \mathcal{Q}$

Gleason's Theorem

States correspond to σ-additive $s : \mathcal{Q} \to [0, 1]$

Wigner's Theorem

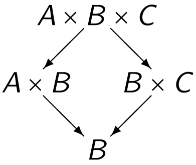
Unitary and anti-unitary maps of $\mathcal{H}$ correspond to auto's of $\mathcal{Q}$

Program

- Try to replace $\mathcal{H}$ with an object X of a suitable category $\mathcal{C}$
- To build an omp $\mathcal{Q}$ from X.
- To use this as a basis of developing aspects of QM.

The categorical setting

Sufficient conditions on the category $\mathcal{C}$ are that it has biproducts, or that it is "honest", meaning it has finite products, projections are epic, and each diagram below is a pushout.



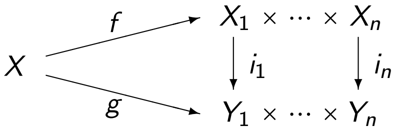
These properties lift to the category $\mathcal{C}^G$ of group representations.

Set, Group, Top, Graphs, G-Sets, etc. are honest.

Constructing $\mathcal{Q}$

Definition An n-ary product map is an iso $f : X \longrightarrow X_1 \times \cdots \times X_n$.

Definition Two such maps are equivalent if there are iso's $i_1, \ldots, i_n$ making the following diagram commute.



Definition An n-ary decomposition of X is an equivalence class

$$[X \simeq_f X_1 \times \cdots \times X_n]$$

Constructing $\mathcal{Q}$

Definition $\mathcal{Q}(X)$ is all binary decompositions $[X \simeq X_1 \times X_2]$ with

1. $0 = [X \simeq \{*\} \times X]$
2. $1 = [X \simeq X \times \{*\}]$
3. $\perp$ is the operation $[X \simeq X_1 \times X_2]^\perp = [X \simeq X_2 \times X_1]$
4. $\leq$ is the relation $[X \simeq X_1 \times (X_2 \times X_3)] \leq [X \simeq (X_1 \times X_2) \times X_3]$

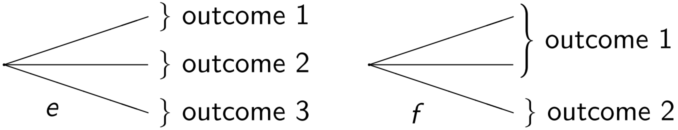
Theorem $\mathcal{Q}(X)$ is an omp.

Note: if $|X| = 4$, then $\mathcal{Q}(X)$ is height 2 with 8 elements.

Physical interpretation

Suppose $e : X \to X_1 \times X_2 \times X_3$ is a ternary product.

Then $f : A \to (X_1 \times X_2) \times X_3$ is a binary product.



Think of n-ary decompositions as experiments with n outcomes.

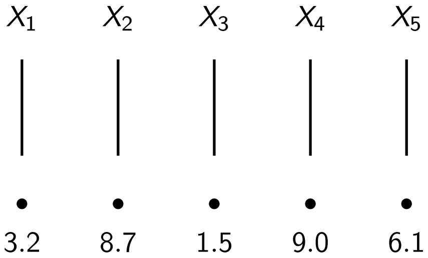
The Spectral Theorem

- Call n-ary decompositions $X \simeq X_1 \times \cdots \times X_n$ n-ary experiments.
- Members of $\mathcal{Q}$ are binary experiments or questions.
- Finite Boolean $\mathcal{B} \leq \mathcal{Q}$ correspond to n-ary experiments.
- Arbitrary $\mathcal{B} \leq \mathcal{Q}$ correspond to sheaf rep's of X in good cases.

The Spectral Theorem

A Boolean subalgebra of $\mathcal{Q}$ consists of compatible questions that can be asked simultaneously such as “is it here” or “here”. We call a Boolean subalgebra of $\mathcal{Q}$ an observable quantity.

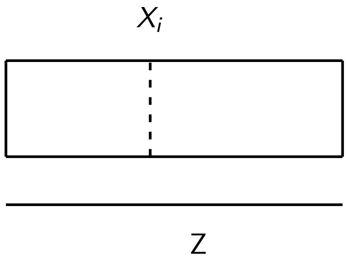
In the finite case, to assign numbers to observable quantity we give a numerical value to each outcome.



The Spectral Theorem

Let $\mathcal{B}$ be an infinite Boolean subalgebra of $\mathcal{Q}$ with Stone space Z.

A scaling of $\mathcal{B}$ is a measurable map $\varphi : \mathsf{Z} \to \mathbb{R} \cup \{\pm\infty\}$.



An observable is an observable quantity together with its scaling. We obtain a calculus of compatible observables with $A + B$, A^2, e^A as in the Hilbert space setting.

States

A state is a (σ) additive map $s : \mathcal{Q} \to [0, 1]$

Theorem Let $\mathcal{B}$ be an observable quantity with Stone space Z scaled by φ. Each state s gives a measure μ_s on Z. Set

$$\mu_s(\varphi^{-1}(U)) = \text{the probability of an outcome in } U$$

$$\int \varphi d\mu_s = \text{the expected value}$$

Note: In standard Hilbert space QM, observables and states arise the same way, but with additional conditions.

Dynamics

Let $\mathcal{C}^G$ be the category of group representations where $G = (\mathbb{R}, +)$.

Objects are pairs (X, E) where X is an object of $\mathcal{C}$ and E is a group homomorphism called the "natural frequency"

$$E : \mathbb{R} \to \text{Aut } X$$

For a Hilbert space $\mathcal{H}$ the standard choice of natural frequency is

$$E(t)v = e^{it}v$$

Dynamics

The Hamiltonian H is an observable of $\mathcal{Q}(X, E)$ associated with a finite scaling $\lambda_1, \dots, \lambda_n$ and decomposition

$$(X, E) \simeq (X_1, E_1) \times \cdots \times (X_n, E_n)$$

Let E^H be the 1-parameter group of automorphisms of X given by

$$E^H(t) = E_1(\lambda_1 t) \times \cdots \times E_n(\lambda_n t)$$

Then the dynamical group U of (X, E) is given by the generalized time independent Schrödinger equation

$$U = E^H$$

Intuitively, pieces at higher energy vibrate more quickly.

Dynamics

Treating Hamiltonians H given by an arbitrary observable rather than a finite one requires more structure on the objects of $\mathcal{C}$.

In the Hilbert space setting the action of the unitary group U for a Hamiltonian H can be approximated within ϵ for all $v \in \mathcal{H}$ and all $t \in [-T, T]$ by some finite observable H' as Hamiltonian.

Tensor products

For systems with structures X_1, X_2 and logics $\mathcal{Q}_1, \mathcal{Q}_2$ we want a structure X for the compound system so that for its logic $\mathcal{Q}$:

1. There is $f : \mathcal{Q}_1 \times \mathcal{Q}_2 \to \mathcal{Q}$
2. This f preserves orthogonal joins in each argument
3. For states σ_i of $\mathcal{Q}_i$, there is a state ω of $\mathcal{Q}$ with

$$\omega(f(q_1, q_2)) = \sigma_1(q_1)\sigma_2(q_2)$$

Tensor products

Let $\mathcal{C}$ be a dagger symmetric monoidal category with biproducts.

$\mathcal{Q}(X)$ is the set of biproduct decompositions of X. Its elements correspond to orthogonal projections $p : X \to X$.

Scalars of $\mathcal{C}$ are morphisms $\mathcal{C}(I, I)$ from the tensor unit I to itself. A scalar s is positive if $s = s^\dagger$ and $s \le t$ if $s + r = t$ for a positive r.

$$[0,1]_{\mathcal{C}} = \{s \in \mathcal{C}(I, I) \mid 0 \le s \le 1\}$$

Tensor products

An element $u : I \to X$ is normal if $u^{\dagger} u = 1$.

Each normal $u : I \to X$ gives a pseudo-state $\sigma_u : \mathcal{Q} \to [0,1]_{\mathcal{C}}$ where

$$\sigma_v(p) = u^{\dagger} p u$$

If the interval unit interval $[0,1]_{\mathcal{C}}$ is contained in real unit interval, these become actual states of the omp $\mathcal{Q}$.

Tensor products

Let $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}$ be omps of decompositions of X, Y and $X \otimes Y$. Define $f : \mathcal{Q}_1 \times \mathcal{Q}_2 \to \mathcal{Q}$ by

$$f(p, q) = p \otimes q$$

If $u : I \to X$ and $v : I \to Y$ are normal, so is $u \otimes v$ and

$$\sigma_{u \otimes v}(p, q) = \sigma_u(p)\sigma_v(q)$$

So the tensor $\otimes$ of $\mathcal{C}$ lifts to a tensor of the omps $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

Concluding remarks

Current work is to carry out this program in specific cases, such as for the categories of sets, normed groups or vector bundles. One example is a Wigner type theorem for sets.

Theorem For X an infinite set, Aut $\mathcal{Q}(X) \simeq$ Aut X.

Mackey's theorem shows that under mild assumptions, any theory of quantum mechanics will produce an omp of questions $\mathcal{Q}$ for each quantum system.

To me, decompositions provide a geometric root for omps.

Thank You