COMPLETIONS OF PSEUDO ORDERED SETS

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ABSTRACT. A pseudo ordered set $(X, \leq)$ is a set X with a binary relation $\leq$ that is reflexive and antisymmetric. We associate to a pseudo ordered set X, a partially ordered set $\Gamma(X)$ called the covering poset. Taking any completion (C, f) of the covering poset $\Gamma(X)$, and a special equivalence relation θ on this completion, yields a completion C/θ of the pseudo ordered set X. The case when (C, f) is the MacNeille completion of $\Gamma(X)$ gives the pseudo MacNeille completion of X.

1. INTRODUCTION

In 1971, Skala [11] introduced the notions of pseudo ordered sets, and trellises. A pseudo ordered set is a set with a relation that is reflexive, and transitive; and a trellis is a pseudo ordered set where any two elements have a greatest lower bound, and a least upper bound. Trellises, in a different signature, are the weakly associate lattices introduced by Fried [3]. Skala proved that every trellis can be order embedded into a complete trellis, that is, into a trellis where each subset has a greatest lower bound and a least upper bound. While Skala states this for trellises, her results show that any pseudo ordered set has an order embedding into a complete trellis.

In this note, for each pseudo ordered set X we construct a poset $\Gamma(X)$, called the covering poset of X, and show that X is a quotient of its covering poset by what we term a convex bounded relation. This is a relation in which each equivalence class is convex and has a least and largest element. It is shown that for any order embedding of the poset $\Gamma(X)$ into a complete lattice C, there is a quotient of C by a convex bounded relation that is a complete trellis into which X is order embedded. Thus, each of the many methods to complete posets provides a corresponding method to complete trellises. Using the MacNeille completion of the poset $\Gamma(X)$ gives what we call the pseudo MacNeille completion of the trellis X. This completion has attractive features, and reduces to the ordinary MacNeille completion when applied to a poset.

In her monograph [12], Skala showed that each pseudo ordered set can be order embedded into a complete trellis. However, Skala's method of completion did not have such attractive features. For example, when applied to a chain, it need not yield even a lattice. Gladstein [6] gave a characterization of when a trellis of finite length is complete. This path has been followed by Bhatta and Shashirekha [8, 9] to give conditions for completeness of arbitrary trellises, similar in spirit to the result of Dilworth that a lattice is complete if and only if each chain has a least upper bound. Otherwise, there seems to be little known about complete trellises and completions of pseudo ordered sets.

This note is arranged in the following way. In the second section, we give background. In the third section, we construct the covering poset $\Gamma(X)$ of X, and in the fourth section we describe a general method for completing pseudo ordered sets. In the fifth section, we provide an abstract characterization of the pseudo MacNeille completion of a pseudo ordered set X. This note has been adapted from the Ph.D. thesis of the first listed author [2].

2. Preliminaries

Around 1970, Skala and Fried independently began to work with generalizations of partial orders and lattices where the underlying order was not required to be transitive. For Skala, [11, 12] these not necessarily transitive generalizations of posets and lattices were called pseudo orders and trellises, respectively. Fried [3] considered not necessarily transitive generalizations of lattices from a universal algebraic viewpoint under what eventually became known as weakly associative lattices. For a guide to some literature on weakly associated lattices, see [5, p. 313].

Definition 2.1. A relation $\leq$ on a set X is a *pseudo order* if for any $a \in X$, $a \leq a$ and if for any $a, b \in X$, $a \leq b$ and $b \leq a$ implies $a = b$. We call $(X, \leq)$ a *pseudo ordered set* if $\leq$ is a pseudo order on X.

The notion of a pseudo ordered set is a very general one. Any poset is a pseudo ordered set. A *tournament* is an assignment of a direction to each edge of a complete graph. Tournaments are exactly those pseudo ordered sets that are totally ordered in that they satisfy $a \leq b$ or $b \leq a$ for all a, b. In fact, pseudo ordered sets obviously correspond to looped directed graphs that contain no 2-cycles.

Definition 2.2. If B is a subset of a pseudo ordered set X, an element c of X is called an *upper bound* of B if $b \leq c$ for every $b \in B$; c is called a *least upper bound* (join) of B if c is an upper bound of B and $c \leq d$ for any upper bound d of B. *Lower bounds* and *greatest lower bounds* (meets) are defined dually. We often write the join of B as $\bigvee B$ and the meet of B as $\bigwedge B$.

Definition 2.3. A *trellis* is a pseudo ordered set $(X, \leq)$ in which any two elements have a least upper bound and a greatest lower bound. We say that X is a *complete trellis* if every subset has a least upper bound and a greatest lower bound.

Trellises have an alternate description [4, 11] as algebras $(X, \wedge, \vee)$ with two idempotent, commutative binary operations that satisfy the absorption laws and the weak associative laws

$$((x \wedge z) \vee (y \wedge z)) \vee z = z \quad \text{and} \quad ((x \vee z) \wedge (y \vee z)) \wedge z = z.$$

Aspects of trellises may be counterintuitive to those used to working with lattices. In a trellis, we need not have $a \wedge b \leq a \vee b$, and as the following example shows, there are finite trellises that are not complete.

Example 2.4. The three element cycle $Z = (\{a, b, c\}, \leq)$ in which $a < b < c < a$ is a trellis, but not a complete trellis.

Definition 2.5. Let X be a pseudo ordered set and $A \subseteq X$.

(1) The set of *lower bounds* of P is $L(A) = \{a \in X : a \leq p \text{ for all } p \in A\}$.
(2) The set of *upper bounds* of P is $U(A) = \{a \in X : p \leq a \text{ for all } p \in A\}$.
(3) We say A is a *normal ideal* if $LU(A) = A$.

In the case that $A = \{a\}$ for some $a \in X$ we use $L(a)$ for $L(A)$ and $U(a)$ for $U(A)$.

While these definitions are familiar from posets, care must be taken because many familiar properties no longer hold in the setting of pseudo ordered sets. For instance, it need not be the case that $LL(A) = L(A)$ or that $LU(a) = L(a)$. However, even in the setting of a pseudo order the pair L, U is a Galois connection since these are the polars of the order relation. In particular (a) L, U are order inverting, (b) $A \subseteq LU(A)$, $A \subseteq UL(A)$, and (c) $LUL = L$, $ULU = U$.

Lemma 2.6. *If every subset of a pseudo ordered set X has a least upper bound then X is a complete trellis.*

Proof. Let $A \subseteq X$. Then $L(A)$ has a least upper bound $\ell = \bigvee L(A)$. For $b \in A$, we have $\ell \leq b$ since b is an upper bound of $L(A)$. So $\ell \in L(A)$. But if $a \in L(A)$, then $a \leq \ell$. Thus $\ell = \bigwedge A$. $\square$

An equivalence relation θ on a pseudo ordered set has an *associated relation* $\trianglelefteq$ on the quotient given by $a/\theta \trianglelefteq b/\theta$ if there exist $x\theta a$ and $y\theta b$ with $x \leq y$. The relation θ is a *pseudo congruence* if the existence of $x, x'\theta a$ and $y, y'\theta b$ with $x \leq y$ and $y' \leq x'$ imply $a\theta b$. It is easily seen that the quotient of a pseudo ordered set by a pseudo congruence is a pseudo order and that $a \leq b$ implies $a/\theta \trianglelefteq b/\theta$. We shall apply these ideas only in a special situation in which the descriptions have a form that is particularly useful for computations.

Definition 2.7. An equivalence relation θ on a poset P is *bounded* if each equivalence class a/θ has a least element $(a/\theta)^l$ and a largest element $(a/\theta)^u$.

The proof of the following is routine.

Proposition 2.8. *Let θ be a bounded equivalence relation on a poset P and $\trianglelefteq$ be its associated relation. Then*

(1) *$a \leq b$ implies $a/\theta \trianglelefteq b/\theta$.*
(2) *$a/\theta \trianglelefteq b/\theta$ iff $(a/\theta)^l \leq (b/\theta)^u$.*
(3) *θ is a pseudo congruence iff $(a/\theta)^l \leq (b/\theta)^u$ and $(b/\theta)^l \leq (a/\theta)^u$ imply $a\theta b$.*

Any lattice congruence is a pseudo congruence, but a lattice congruence even on a complete lattice need not be bounded. The following extends the known result that the quotient of a complete lattice by a bounded lattice congruence is again a complete lattice.

Proposition 2.9. *If P is a complete poset and θ is a bounded pseudo congruence on P, then the quotient $(P/\theta, \trianglelefteq)$ is a complete trellis.*

Proof. Given $\mathcal{D} \subseteq P/\theta$, let $D = \{(a/\theta)^l : a/\theta \in \mathcal{D}\}$. For $a/\theta \in \mathcal{D}$, we have $(a/\theta)^l \leq \bigvee D$, so $a/\theta = (a/\theta)^l/\theta \trianglelefteq (\bigvee D)/\theta$. Therefore $(\bigvee D)/\theta$ is an upper bound of $\mathcal{D}$. Suppose b/θ is an upper bound of $\mathcal{D}$. Then for $a/\theta \in \mathcal{D}$ we have $(a/\theta)^l \leq (b/\theta)^u$, so $\bigvee D \leq (b/\theta)^u$. This implies that $(\bigvee D)/\theta \trianglelefteq (b/\theta)$. So $(\bigvee D)/\theta$ is the least upper bound of $\mathcal{D}$. Greatest lower bounds are established dually. $\square$

We next adapt the notions of completions of posets to the setting of pseudo ordered sets in an obvious way. To begin, a map $f : X \to Y$ between pseudo ordered sets is *order preserving* if $a \leq b$ implies $f(a) \leq f(b)$. We say that f is an *order embedding* if $a \leq b$ iff $f(a) \leq f(b)$, and an *order isomorphism* if it is an order embedding and onto. As is common, we denote the pseudo orders on X and Y by the same symbol since confusion is not likely.

Definition 2.10. An *extension* of a pseudo ordered set X is a pair (E, f) where E is a pseudo ordered set and $f : X \to E$ is an order embedding. If E is a complete trellis, we say that the extension (E, f) is a *completion* of X.

It is well known that every poset can be embedded into a complete lattice. Skala [11, 12] gave a result showing that every trellis can be order embedded into a complete trellis. In fact, Skala's proof showed more, that every pseudo ordered set can be order embedded into a complete trellis, that is, that every pseudo ordered set has a completion. To describe Skala's completion, we first need the following.

Definition 2.11. For any subset A of a pseudo ordered set X we define the *closure* of A, denoted as A^*, to be the intersection of all subsets Q of X satisfying the following:

(1) $A \subseteq Q$
(2) if $Q' \subseteq Q$ and $\bigvee Q'$ exists, then $\bigvee Q'$ is in Q.

Thus A^* is the smallest set that contains A and is closed under existing joins.

Definition 2.12 (Skala's Completion)**.** For a pseudo ordered set X, let $B = X \cup S^*$ where $S = \{A \subseteq X : A \text{ has no join}\}$ and $S^* = \{A^* : A \in S\}$. The pseudo order $\trianglelefteq$ on B is given as follows. Here $x, y \in X$ and $P, Q \in S^*$.

(1) $x \trianglelefteq y$ iff $x \leq y$
(2) $x \trianglelefteq P$ iff x is in P
(3) $P \trianglelefteq x$ iff $z \leq x$ for every z in P
(4) $P \trianglelefteq Q$ iff $P \subseteq Q$

In order to distinguish between joins and meets in X and in B, we use $\bigvee$ and $\bigwedge$ for joins and meets in X, and Σ and Π for joins and meets in B. Skala established the following [11].

Proposition 2.13. *For $\mathcal{C} \subseteq B$ let $D = \{x : x \in X \text{ and } x \in \mathcal{C}\} \cup \{x : x \in C \text{ for some } C \in \mathcal{C}\}$. Then*

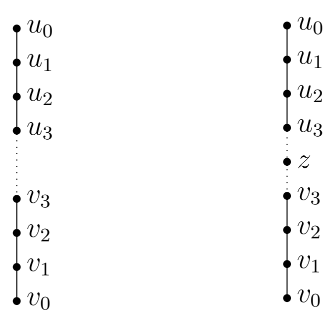
$$\Sigma\, \mathcal{C} = \begin{cases} \bigvee D & \textit{if it exists;} \\ D^* & \textit{otherwise.} \end{cases}$$

Then by Lemma 2.6, we have the result.

Theorem 2.14. *For a pseudo ordered set X, the pair (B, f) is a completion of X, where B is the pseudo ordered set constructed in Definition 2.12 and f is the inclusion map.*

While Skala's completion shows that every pseudo ordered set can be order embedded into a complete trellis, it is artificial, and quite poorly behaved in nearly every other aspect.

Example 2.15. Skala's completion applied to a chain X.



X MacNeille completion of X

Consider the chain X shown at left. This is a bounded lattice, but is not complete. Its MacNeille completion is the complete lattice shown at right. It is also a chain. To describe Skala's completion $B = X \cup S^*$ of X, we note that S is the set of subsets of X that have no join. Thus S is the set of infinite subsets of $V = \{v_n : n \in \mathbb{N}\}$. Since v_0 is the join of the empty set, the set S^* of closures of members of S is the set of infinite subsets of V that contain v_0. The pseudo order of B is not even a partial order. Indeed, for $P = \{v_{2n} : n \in \mathbb{N}\}$ we have $P \in S^*$. Then $v_1 \trianglelefteq v_2$, and $v_2 \trianglelefteq P$ since $v_2 \in P$, but it is not the case that $v_1 \trianglelefteq P$.

Remark 2.16. One might ask why Skala looked to unusual methods to complete pseudo ordered sets. One can complete a poset in well known and elegant ways. For a poset P, the collection $\mathcal{D}(P)$ of its downsets, or $M(P)$ of its normal ideals, are complete lattices; and there is an order embedding of P into these complete lattices that in both cases is given by $a \rightsquigarrow L(a)$. For pseudo orders, the notion of a downset is problematic since the ordering is not transitive. However, the notion of a normal ideal remains intact, and as the pair L, U is a Galois connection on a pseudo ordered set, the collection of all normal ideals forms a complete lattice under set inclusion. But herein lies the problem. One has no hope of finding an order embedding of pseudo ordered set with a non-transitive relation into a complete lattice, not even into a poset whose order relation is necessarily transitive. For a pseudo ordered set, this failure is realized by the observation that $L(a) \subseteq L(b) \Rightarrow a \leq b$, but not conversely.

3. The Covering Poset

It is obvious that to each pseudo ordered set $(X, \leq)$, one can associate a poset. One repairs the defect of lack of transitivity in the order $\leq$ by taking its transitive closure tr $(\leq)$. The result is reflexive and transitive, but perhaps antisymmetry is lost. But tr $(\leq)$ is a quasi order, so gives rises to an equivalence relation θ and the quotient X/θ then naturally carries a partial order. Further, the canonical quotient map $x \rightsquigarrow x/\theta$ is order preserving. However, all this is of little use in obtaining a completion of X for a variety of reasons, including that a completion must be an embedding. The idea here is a different one. Rather than finding a poset that is a quotient of X, we find a poset that has X as a quotient.

Definition 3.1. Let X be a pseudo ordered set. For an element $a \in X$, we say that a is *transitive* if for all $b, c \in X$, we have that $b \leq a \leq c$ implies $b \leq c$.

Definition 3.2. Let X be a pseudo ordered set, T be the set of its transitive elements, and N be the set of its elements that are not transitive. Let $\Gamma(X)$ be the set $(N \times \{0, 1\}) \cup T$. Define unary operations $+, -$ on $\Gamma(X)$ in the following way. For $a \in X$:

(1) if $a \in N$ then $a^+ = (a, 1)$ and $a^- = (a, 0)$,
(2) if $a \in T$ then $a^+ = a = a^-$.

Note that from the above definition, each element x of $\Gamma(X)$ can be expressed as a^+ or a^- for some $a \in A$. The choice of $a \in A$ is unique, but some elements $a \in \Gamma(X)$ are given by both a^+ and a^- for some $a \in X$, and this happens exactly when a is transitive.

Definition 3.3. Let $\sqsubseteq$ be the binary relation on $\Gamma(X)$ given by:

(1) $a^+ \sqsubseteq b^+$ iff $\ell \leq a$ implies $\ell \leq b$.
(2) $a^+ \sqsubseteq b^-$ iff $\ell \leq a$ and $b \leq u$ implies $\ell \leq u$.
(3) $a^- \sqsubseteq b^+$ iff $a \leq b$.
(4) $a^- \sqsubseteq b^-$ iff $b \leq u$ implies $a \leq u$.

One must take care to note that if $a^+ = a^-$, or if $b^+ = b^-$, then there is no conflict in the definition. This is a simple consequence of the observation that in such case a or b is transitive. In fact, a is transitive iff $a^- = a^+$ iff $L(a) = LU(a)$. To work with the relation $\sqsubseteq$, it is convenient to reformulate it in terms of the operations L, U of taking lower bounds and upper bounds. For this, recall that L, U are a Galois connection on X.

Lemma 3.4. *For X a pseudo ordered set and $\sqsubseteq$ the associated relation on $\Gamma(X)$, for all $a, b \in X$ we have the following.*

(1) $a^+ \sqsubseteq b^+$ *iff* $L(a) \subseteq L(b)$
(2) $a^+ \sqsubseteq b^-$ *iff* $L(a) \subseteq LU(b)$
(3) $a^- \sqsubseteq b^+$ *iff* $LU(a) \subseteq L(b)$
(4) $a^- \sqsubseteq b^-$ *iff* $LU(a) \subseteq LU(b)$

Proof. Statements (1) and (2) are immediate from the definitions. For (3), if $a^- \sqsubseteq b^+$ then $a \leq b$. So $\{a\} \subseteq L(b)$. This gives $LU(a) \subseteq LUL(b) = L(b)$. Conversely, since $a \in LU(a)$, if $LU(a) \subseteq L(b)$, then $a \in L(b)$, so $a \leq b$, hence $a^- \sqsubseteq b^+$. For (4), the definition of $a^- \sqsubseteq b^-$ is equivalent to $\{a\} \subseteq LU(b)$. This implies that $LU(a) \subseteq LULU(b) = LU(b)$. Conversely, since $a \in LU(a)$, if $LU(a) \subseteq LU(b)$, then $\{a\} \subseteq LU(b)$, hence $a^- \sqsubseteq b^-$. □

Proposition 3.5. *For a pseudo ordered set X, the relation $\sqsubseteq$ on $\Gamma(X)$ is a partial ordering and the poset $\Gamma(X)$ is isomorphic to the poset whose underlying set is $\mathcal{N} = \{L(a), LU(a) : a \in X\}$ and whose partial ordering is set inclusion.*

Proof. We begin by showing that $\sqsubseteq$ is antisymmetric. From the symmetry inherent in the definition, it is enough to consider the cases (a) $a^+ \sqsubseteq b^+$, $b^+ \sqsubseteq a^+$, and (b) $a^- \sqsubseteq b^+$, $b^+ \sqsubseteq a^-$. In case (a), 3.4.1 gives $L(a) = L(b)$. Since $a \in L(a)$ we then have $a \leq b$, and similarly $b \leq a$. Then $a = b$, so $a^+ = b^+$. In case (b) 3.4.2 and 3.4.3 give $LU(a) \subseteq L(b) \subseteq LU(a)$. So $LU(a) = L(b)$. This then gives $a = b$, and therefore $L(a) = LU(a)$. This implies that a is transitive, so $a^- = b^+$.

Next, define $f : \Gamma(X) \to \mathcal{N}$ by setting $f(a^+) = L(a)$ and $f(a^-) = LU(a)$. If $a^- = a^+$ then a is transitive, so $L(a) = LU(a)$. Thus f is well defined. It is clearly onto. For $x, y \in \Gamma(X)$, Lemma 3.4 gives $f(x) \subseteq f(y)$ iff $x \sqsubseteq y$. Then if $f(x) = f(y)$, we have $x \sqsubseteq y$ and $y \sqsubseteq x$, and since $\sqsubseteq$ is antisymmetric, $x = y$. So f is a bijection with $x \sqsubseteq y$ iff $f(x) \subseteq f(y)$. It follows that $\sqsubseteq$ is a partial ordering and f is an order isomorphism between the posets $\Gamma(X)$ and $\mathcal{N}$. □

Example 3.6. The covering poset $\Gamma(Z)$ of three element cycle Z of Example 2.4 is shown in Figure 1. This example illustrates that the covering poset of a trellis is not necessarily a lattice. In general, the covering poset of an n-cycle is an n-crown.

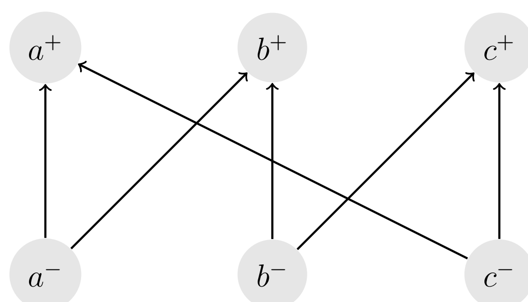


FIGURE 1. The covering poset $\Gamma(Z)$ of the 3-element cycle Z

Definition 3.7. For a pseudo ordered set X, let θ_X be the relation on its covering poset $\Gamma(X)$ given by

$$x\theta_X y \quad \text{iff} \quad x, y \in \{a^-, a^+\} \text{ for some } a \in X.$$

For the following, we suggest the reader review Definition 2.7, the material that precedes it, and the Proposition 2.8 that follows it. We also recall that a subset C of a poset P is *convex* if $a, c \in C$ and $a \leq b \leq c$ imply $b \in C$. An equivalence relation on a poset is *convex* if each of its equivalence classes is convex.

Proposition 3.8. *The relation θ_X is a convex bounded pseudo congruence on $\Gamma(X)$.*

Proof. Since $\{a^-,a^+\}$ and $\{b^-,b^+\}$ are disjoint or equal, it follows that θ_X is an equivalence relation and the sets $\{a^-,a^+\}$ for $a\in X$ are its equivalence classes. Let $a\in X$. Since $a^-\sqsubseteq a^+$ the equivalence class $\{a^-,a^+\}$ has a smallest element $\{a^-,a^+\}^l=a^-$ and a largest element $\{a^-,a^+\}^u=a^+$. So this relation is bounded. If $a^-\sqsubseteq b^+$ and $b^-\sqsubseteq a^+$, then by Definition 3.3 we have $a\leq b$ and $b\leq a$, and as X is a pseudo order $a=b$. So θ_X is a pseudo congruence.

To see that θ_X is convex, suppose $x,y,z\in\Gamma(X)$, $x\theta_X z$, and $x\sqsubseteq y\sqsubseteq z$. We must show $x\theta_X y$. Since $\Gamma(X)$ is a poset, we have $x\sqsubseteq z$, and our conclusion is trivial if $x=z$. So we may assume that $x=a^-$ and $z=a^+$ for some $a\in X$. Assume $y=b^i$ for $i\in\{+,-\}$. Consider first the case that $y=b^+$ so $a^-\sqsubseteq b^+\sqsubseteq a^+$. Then by Definition 3.3.3 we have $a\leq b$, and by Definition 3.3.1 we have $\ell\leq b\Rightarrow\ell\leq a$. But $b\leq b$, so this implies that $b\leq a$, hence $a=b$. Then $x=a^-$ and $y=a^+$, and this implies that $x\theta_X y$. In the case $y=b^-$, then $a^-\sqsubseteq b^-\sqsubseteq a^+$, so Definition 3.3.3 gives $b\leq a$. Then Definition 3.3.4 gives $a\leq b$. □

Definition 3.9. For a pseudo ordered set X, its *covering map* $c_X:\Gamma(X)\to X$ is given by setting $c_X(a^-)=a$ and $c_X(a^+)=a$ for $a\in X$.

Proposition 3.10. *The covering map $c_X:\Gamma(X)\to X$ is order preserving.*

Proof. Consider the four cases of Definition 3.3. For 3.3.1, if $a^+\sqsubseteq b^+$, then using $\ell=a$, since $a\leq a$, we have $a\leq b$. The other cases are similar. □

By Proposition 3.8, we have that θ_X is a pseudo congruence on $\Gamma(X)$. So the associated relation $\trianglelefteq$ on the quotient $\Gamma(X)/\theta_X$ is a pseudo order. The equivalence classes of θ_X are the sets $\{a^-,a^+\}$ for $a\in X$, with a^- its least element and a^+ its largest. So by Proposition 2.8.2 we have $\{a^-,a^+\}\trianglelefteq\{b^-,b^+\}$ iff $a^-\sqsubseteq b^+$. By Definition 3.3.3 this is equivalent to $a\leq b$. This establishes the following.

Theorem 3.11. *Let X be a pseudo ordered set. Then the covering map c_X and canonical quotient map κ_X are order preserving, and the map $\iota_X:\Gamma(X)/\theta_X\to X$ given by $\iota_X(\{a^-,a^+\})=a$ is an order isomorphism with $c_X=\iota_X\circ\kappa_X$.*

$$\begin{tikzcd} \Gamma(X) \arrow[r, "c_X"] \arrow[d, "\kappa_X"'] & X \\ \Gamma(X)/\theta_X \arrow[ur, "\iota_X"'] & \end{tikzcd}$$

4. A general method of completing pseudo ordered sets

In this section, we provide a general method to complete a pseudo ordered set X by completing its covering poset $\Gamma(X)$ then taking a quotient. Throughout, we assume X is a pseudo ordered set, P is a poset and $f:\Gamma(X)\to P$ is an order embedding.

Definition 4.1. Define θ_f on P by

$$x\theta_f y \text{ iff } x=y \text{ or } f(a^-)\leq x,y\leq f(a^+) \text{ for some } a\in A.$$

Proposition 4.2. *The relation θ_f is a convex bounded pseudo congruence and for $a\in X$, the interval $[f(a^-),f(a^+)]$ is an equivalence class of θ_f, and each non-trivial class is of this form.*

Proof. First we show that θ_f is an equivalence relation. The relation is clearly reflexive and symmetric. For transitivity, let $x, y, z \in P$ such that $x\theta_f y$ and $y\theta_f z$. If $x = y$ or $y = z$, clearly $x\theta_f z$. For the remaining possibility, $f(a^-) \leq x, y \leq f(a^+)$ and $f(b^-) \leq y, z \leq f(b^+)$ for some $a, b \in X$. Then $f(a^-) \leq f(b^+)$ and $f(b^-) \leq f(a^+)$. Since f is an order embedding, we have $a^- \sqsubseteq b^+$ and $b^- \sqsubseteq a^+$, which imply $a \leq b$ and $b \leq a$. So, $a = b$. Then $f(a^-) \leq x, y, z \leq f(a^+)$. Therefore, $x\theta_f z$.

We next show that x/θ_f has a least and largest element. If $x/\theta_f = \{x\}$ clearly x is the least and largest element. Suppose there is $y \neq x$ with $y \in x/\theta_f$. Then there is $a \in X$ such that $f(a^-) \leq x, y \leq f(a^+)$. If $z \in x/\theta_f$ with $z \neq x$ there is a $b \in X$ such that $f(b^-) \leq x, z \leq f(b^+)$. So $f(a^-) \leq x \leq f(b^+)$ implies $a^- \sqsubseteq b^+$. Hence $a \leq b$. And $f(b^-) \leq x \leq f(a^+)$, so $b^- \sqsubseteq a^+$. Then $b \leq a$. So $a = b$. Therefore for any $y \in x/\theta_f$ we have that $f(a^-) \leq y \leq f(a^+)$. Clearly $f(a^-), f(a^+)$ are in x/θ_f. Then $f(a^-) = (x/\theta_f)^l$ and $f(a^+) = (x/\theta_f)^u$.

To see that x/θ_f is convex, note that this is trivial if $x/\theta_f = \{x\}$. If there is $y \neq x$ with $y \in x/\theta_f$, then there is $a \in X$ such that each element of x/θ_f lies in the interval $[f(a^-), f(a^+)]$. But by the definition of θ_f, each element in this interval is θ_f-related to x. Thus x/θ_f is equal to this interval, and hence is convex.

Since θ_f is a bounded, to show it is a pseudo congruence, it is sufficient by Proposition 2.8.3 to show that $(x/\theta_f)^l \leq (y/\theta_f)^u$ and $(y/\theta_f)^l \leq (x/\theta_f)^u$ imply that $x\theta_f y$. If $x/\theta_f = \{x\}$, the assumptions give $(y/\theta_f)^l \leq x, y \leq (y/\theta_f)^u$. Then as θ_f is convex and bounded, this gives $x\theta_f y$. The situation is similar if $y/\theta_f = \{y\}$. In the remaining case, there are $a, b \in X$ with $f(a^-) = (x/\theta_f)^l$, $f(a^+) = (x/\theta_f)^u$, $f(b^-) = (y/\theta_f)^l$ and $f(b^+) = (y/\theta_f)^u$. The assumptions then give $f(a^-) \leq f(b^+)$ and $f(b^-) \leq f(a^+)$. As we have argued several times, this gives $a = b$. Then by the definition of θ_f, or by the fact that it is bounded and convex, we have $x\theta_f y$. □

Using Propositions 2.8, 2.9 and 4.2 we have the following.

Corollary 4.3. *If P is a complete lattice, then P/θ_f is a complete trellis whose ordering $\trianglelefteq$ is given by*

$$x/\theta_f \trianglelefteq y/\theta_f \quad \text{iff} \quad (x/\theta_f)^l \leq (y/\theta_f)^u.$$

From a pseudo ordered set X, we construct its covering poset $\Gamma(X)$. Then for an order embedding $f : \Gamma(X) \to P$ into a complete lattice P, that is, for a completion in the ordinary sense of the poset $\Gamma(X)$, we obtain a complete trellis P/θ_f. Of course, we seek a completion of the pseudo ordered set X. For this, we require an order embedding of X into P/θ_f.

Definition 4.4. Let $g_f : X \to P/\theta_f$ be given by $g_f(a) = f(a^+)/\theta_f$.

Theorem 4.5. *For a completion $f : \Gamma(X) \to P$ of the poset $\Gamma(X)$ into a complete lattice, $g_f : X \to P/\theta_f$ is a completion of the trellis X. Further, for the covering map c_X and canonical quotient map κ_f, the following diagram commutes.*

$$\begin{tikzcd}
\Gamma(X) \arrow[r, "f"] \arrow[d, "c_X"'] & P \arrow[d, "\kappa_f"] \\
X \arrow[r, "g_f"] & P/\theta_f
\end{tikzcd}$$

Proof. Let $a, b \in X$. By Definition 3.3.3 we have $a \leq b$ iff $a^- \sqsubseteq b^+$. Since f is an order embedding, this is equivalent to $f(a^-) \leq f(b^+)$. By Proposition 4.2, $f(a^-) = (f(a^+)/\theta_f)^l$ and $f(b^+) = (f(b^+)/\theta_f)^u$. So by Corollary 4.3, $a \leq b$ is equivalent to $f(a^+)/\theta_f \trianglelefteq f(b^+)/\theta_f$, and hence to $g_f(a) \trianglelefteq g_f(b)$. So g_f is an order embedding. It is trivial that the diagram commutes. $\square$

There are several additional properties of this method of completing a pseudo order that are of interest. In the following, we use $\operatorname{Im} f$ for the image of a function f.

Definition 4.6. An order embedding $f : A \to C$ between pseudo ordered sets is *strict* if it satisfies the following. If $n \geq 1$ and $c_1, c_2, ..., c_n \in C$ with $f(a) \leq c_1 \leq \cdots \leq c_n \leq f(b)$, then either $c_i \in \operatorname{Im} f$ for some $i \leq n$ or $a \leq b$.

We use the terms *strict extension* and *strict completion* with the obvious meaning.

Proposition 4.7. *If $f : \Gamma(X) \to P$ is an order embedding of the poset $\Gamma(X)$ into a complete lattice P, then the completion $g_f : X \to P/\theta_f$ is strict.*

Proof. Let $a, b \in X$, $n \geq 1$ and $p_1, ..., p_n \in P$ with $g_f(a) \trianglelefteq p_1/\theta_f \trianglelefteq \cdots \trianglelefteq p_n/\theta_f \trianglelefteq g_f(b)$. Suppose that $p_i/\theta_f \notin \operatorname{Im} g_f$ for all $i \leq n$. Then by Proposition 4.2, each equivalence class p_i/θ_f is trivial for each $i \leq n$. From the description of $\trianglelefteq$ in Corollary 4.3 and the description of the largest and least members of the equivalence classes θ_f, we have $f(a^-) \leq p_1 \leq \cdots \leq p_n \leq f(b^+)$. Since P is a complete lattice, its order is transitive. So $f(a^-) \leq f(b^+)$. Since f is an order embedding, then $a^- \sqsubseteq b^+$. Then by Definition 3.3.3 we have $a \leq b$. $\square$

The following result shows that the diagram in Theorem 4.5 has a universal property.

Proposition 4.8. *Suppose Q is a pseudo ordered set and $u : P \to Q$ and $v : X \to Q$ are order preserving with $v \circ c_X = u \circ f$. Then there exists a unique order preserving map $h : P/\theta_f \to Q$ with $h \circ \kappa_f = u$ and $h \circ g_f = v$.*

$$\begin{tikzcd}
\Gamma(X) \arrow[r, "f"] \arrow[d, "c_X"'] & P \arrow[d, "\kappa_f"'] \arrow[ddr, "u"] & \\
X \arrow[r, "g_f"] \arrow[drr, "v"'] & P/\theta_f \arrow[dr, dashed, "h"] & \\
& & Q
\end{tikzcd}$$

Proof. Define $h(x/\theta_f) = u(x)$. To see that h is well defined, let $x, y \in P$ with $x\theta_f y$. Then either $x = y$ or $f(a^-) \leq x, y \leq f(a^+)$ for some $a \in X$. If $x = y$ clearly $u(x) = u(y)$. Otherwise, $u \circ f(a^-) = v \circ c_X(a^-) = v(a)$ and $u \circ f(a^+) = v \circ c_X(a^+) = v(a)$. So, $u \circ f(a^-) = u \circ f(a^+)$. Since u is order preserving, $u \circ f(a^-) \leq u(x), u(y) \leq u \circ f(a^+)$. So $u(x) = u(y)$.

Also for each $x \in P$ and $a \in X$, we have $h \circ \kappa_f(x) = h(x/\theta_f) = u(x)$. So $h \circ k = u$. And $h \circ g_f(a) = h(f(a^+)/\theta_f) = u \circ f(a^+) = v \circ c_X(a^+) = v(a)$. So $h \circ g_f = v$.

To see that h is an order preserving map, let x/θ_f and y/θ_f be such that $x/\theta_f \trianglelefteq y/\theta_f$. By Corollary 4.3, $(x/\theta_f)^l \leq (y/\theta_f)^u$. We have several cases. For the first case, suppose $(x/\theta_f)^l = x$

and $(y/\theta_f)^u = y$. Then $x \leq y$. Since u is an order preserving map, we have $u(x) \leq u(y)$ this implies $h(x/\theta_f) \leq h(y/\theta_f)$. For the second case, let $(x/\theta_f)^l = f(a^-)$ and $(y/\theta_f)^u = f(b^+)$ for some $a, b \in X$. Then $f(a^-) \leq f(b^+)$. Since u is an order preserving map, $uf(a^-) \leq uf(b^+)$. Since $u = h \circ \kappa_f$, we have $h(f(a^-)/\theta_f) \leq h(f(b^+)/\theta_f)$. In addition we know that $f(a^-)\theta_f x$ and $f(b^+)\theta_f y$, so $h(x/\theta_f) \leq h(y/\theta_f)$. For the third case, assume $f(a^-) = (x/\theta_f)^l$ and $(y/\theta_f)^u = y$ for some $a \in X$. Then $f(a^-) \leq y$. Applying u in both sides gives $uf(a^-) \leq u(y) = h(y/\theta_f)$. Also, $uf(a^-) = h \circ \kappa_f(f(a^-)) = h(x/\theta_f)$ as above, so $h(x/\theta_f) \leq h(y/\theta_f)$. The fourth case is similar to the third. Therefore, h is an order preserving map. The uniqueness of h is given by the commutativity of the lower part of the diagram. □

5. The pseudo MacNeille completion

One of the best known methods to complete a poset is the MacNeille completion. This is also known as the Dedekind completion, normal completion, completion by cuts, among other names. It was introduced by MacNeille as an extension of Dedekind's method of conditionally completing the rationals to the reals by "cuts". A normal ideal of a poset P is a subset $N \subseteq P$ with $N = LU(P)$. The collection of all normal ideals of P is a complete lattice and there is an order embedding of P into it taking a to the principle ideal $L(a)$ it generates. This embedding is both join and meet dense, and it was shown by Banaschewski and Schmidt [1, 10] that these properties characterize the MacNeille completion up to unique commuting isomorphism. In hindsight, it is advantageous to define the MacNeille completion of a poset as a join and meet dense completion, and to use the construction via normal ideals as a means to establish existence. We will follow this path to introduce the pseudo MacNeille completion of a pseudo ordered set and establish its basic properties.

Definition 5.1. Let $f : X \to Y$ be an order preserving map between pseudo ordered sets. We say that f is *join dense* if for each $y \in Y$ we have that $y = \bigvee\{f(x) : f(x) \leq y\}$ and that f is *meet dense* if for each $y \in Y$ we have $y = \bigwedge\{f(x) : y \leq f(x)\}$.

A map $f : X \to Y$ between pseudo ordered sets *preserves existing joins* if for each $S \subseteq X$, if S has a join in X, then its image $f(S)$ has a join in Y and $f(\bigvee S) = \bigvee f(S)$. A similar condition describes when f *preserves existing meets.*

Proposition 5.2. *Let $f : X \to Y$ be a strict order embedding between pseudo ordered sets. If f is meet dense, then it preserves existing joins; and if f is join dense, then it preserves existing meets.*

Proof. We only show the statement that meet dense implies preserving existing joins, the other is dual. Suppose $S \subseteq X$ has a join. Since f is order preserving, $f(\bigvee S)$ is an upper bound of the image $f(S)$. Suppose y is another upper bound of $f(S)$, that is, $f(s) \leq y$ for all $s \in S$. By meet density $y = \bigwedge\{f(x) : y \leq f(x)\}$. So,

$$f(s) \leq y \leq f(x) \text{ for all } s \in S \text{ and for all } x \in X \text{ with } y \leq f(x).$$

Since f is strict, either y is in the image of f, or $s \leq x$ for all $s \in S$ and all $x \in X$ with $y \leq f(x)$.

If y is in the image of f, then there is $z \in X$ with $f(z) = y$. Then $f(s) \leq f(z)$ for all $s \in S$. Since f is an order embedding, $s \leq z$ for all $s \in S$, hence $\bigvee S \leq z$. Therefore $f(\bigvee S) \leq f(z) = y$. If y is not in the image of f, then we have that $f(s) \leq f(x)$ for all $s \in S$ and all $x \in X$ with $y \leq f(x)$. Since f is an order embedding, we have $\bigvee S \leq x$ for each $x \in X$ with $y \leq f(x)$, and therefore $f(\bigvee S) \leq f(x)$ for each $x \in X$ with $y \leq f(x)$. Since $y = \bigwedge\{f(x) : y \leq f(x)\}$, it follows that $f(\bigvee S) \leq y$. □

Definition 5.3. A completion (E, e) of a pseudo ordered set X is called a *pseudo MacNeille completion* of X if $e : X \to X$ is join dense, meet dense, and strict.

We use the article "a" in this definition because at this point we do not know of the existence of a pseudo MacNeille completion, and as there may potentially be many. As we will see later, each pseudo ordered set has a pseudo MacNeille completion that is unique up to unique commuting isomorphism. Once established, the article "the" becomes appropriate, when understood in the correct way.

Lemma 5.4. *Let (E, e) and (F, f) be pseudo MacNeille completions of X. For $g : E \to F$ given by $g(c) = \bigvee\{f(x) : e(x) \leq c\}$*

(1) *$g \circ e = f$.*
(2) *g is order preserving.*
(3) *If $g(c) = f(z)$ then $c = e(z)$.*
(4) *g is an order embedding.*
(5) *g is strict.*

Proof. For (1), let $z \in X$. Then $g(e(z)) = \bigvee\{f(x) : e(x) \leq e(z)\}$. Since $e(x) \leq e(z)$ iff $x \leq z$, which occurs iff $f(x) \leq f(z)$, we have $g(e(z)) = \bigvee\{f(x) : f(x) \leq f(z)\} = f(z)$.

For (2), suppose $a, b \in E$ with $a \leq b$. Suppose first that $a \in \mathrm{Im}(e)$. Then $a = e(z)$ for some $z \in X$ and by (1), $g(a) = f(z)$. Since $g(b) = \bigvee\{f(x) : e(x) \leq b\}$ and $e(z) = a \leq b$, we have $g(a) = f(z) \leq g(b)$. Suppose $a \notin \mathrm{Im}(e)$. We show the following

$$(*) \quad \text{if } x \in X \text{ and } e(x) \leq a \leq b, \text{ then } e(x) \leq b.$$

If $b \in \mathrm{Im}(e)$, this follows since e is strict. Suppose that $b \notin \mathrm{Im}(e)$, then for any $y \in X$, if $e(x) \leq a \leq b \leq e(y)$, then since e is strict and $a, b \notin \mathrm{Im}(e)$ we have $e(x) \leq e(y)$. Since e is meet dense, $b = \bigwedge\{e(y) : b \leq e(y)\}$, and therefore $e(x) \leq b$. This establishes $(*)$. Using $(*)$ we then have $g(a) = \bigvee\{f(x) : e(x) \leq a\} \leq \bigvee\{f(x) : e(x) \leq b\} = g(b)$.

For (3), suppose $g(c) = f(z)$ and let $S = \{x : e(x) \leq c\}$. By definition, $g(c) = \bigvee f(S)$. So $f(x) \leq f(z)$ for all $x \in S$. Since f is an order embedding, z is an upper bound of S, and as e is order preserving we have $e(z)$ is an upper bound of $e(S)$. The join density of e gives $c = \bigvee e(S)$, and therefore $c \leq e(z)$. To see $e(z) \leq c$, by meet density $c = \bigwedge\{e(y) : c \leq e(y)\}$, so it is enough to show that $e(z) \leq e(y)$ for all y such that $c \leq e(y)$. But if $c \leq e(y)$, then since g preserves order and $g \circ e = f$, we have $f(z) = g(c) \leq g(e(y)) = f(y)$, hence $z \leq y$, and so $e(z) \leq e(y)$.

For (4), suppose $g(a) \leq g(b)$ for some $a, b \in E$. Using join and meet density of e we have

$$a = \bigvee\{e(x) : e(x) \leq a\} \qquad b = \bigwedge\{e(y) : b \leq e(y)\}$$

For any $x, y \in X$ with $e(x) \leq a$ and $b \leq e(y)$, since $g \circ e = f$ and g is order preserving,

$$f(x) \leq g(a) \leq g(b) \leq f(y).$$

If $g(a), g(b) \notin \mathrm{Im}(f)$, then as f is strict, $x \leq y$, and hence $e(x) \leq e(y)$, whenever $e(x) \leq a$ and $b \leq e(y)$. It follows from join and meet density that $a \leq b$. If $g(a) \in \mathrm{Im}(f)$ and $g(b) \notin \mathrm{Im}(f)$, then $g(a) = f(z)$ for some $z \in X$. By (3), $a = e(z)$. So for all y with $b \leq e(y)$ we have $f(z) \leq g(b) \leq f(y)$, and since f is strict and $g(b) \notin \mathrm{Im}(f)$, then $z \leq y$, and hence $a = e(z) \leq e(y)$. By meet density, we have $a \leq b$. A similar argument applies if $g(a) \notin \mathrm{Im}(f)$ and $g(b) \in \mathrm{Im}(f)$. Finally, if $g(a), g(b) \in \mathrm{Im}(f)$, then $g(a) = f(z)$ and $g(b) = f(w)$ for some $z, w \in X$. By (3), $a = e(z)$ and $b = e(w)$. Then $g(a) \leq g(b)$ gives $f(z) \leq f(w)$, so $z \leq w$. Hence $a \leq b$.

For (5), suppose that $a, b \in E$ and $w_1, ..., w_n \in F$ with

$$g(a) \leq w_1 \leq \cdots \leq w_n \leq g(b).$$

If any of $w_1, ..., w_n$ belong to $\mathrm{Im}(g)$ there is nothing to show. So suppose $w_1, ..., w_n \notin \mathrm{Im}(g)$. We must show that $g(a) \leq g(b)$, or equivalently, that $a \leq b$. Note that by (1), $\mathrm{Im}(f) \subseteq \mathrm{Im}(g)$, so none of $w_1, \ldots, w_n$ belong to $\mathrm{Im}(f)$. Consider several cases.

Suppose first that $g(a), g(b) \in \mathrm{Im}(f)$. So $g(a) = f(z)$ and $g(b) = f(w)$ for some $z, w \in X$. Then $f(z) \leq w_1 \leq \cdots \leq w_n \leq f(w)$. Since f is strict and $w_1, ..., w_n \notin \mathrm{Im}(f)$, then $z \leq w$, giving $f(z) \leq f(w)$ and hence $g(a) \leq g(b)$.

Next, suppose $g(a) \in \mathrm{Im}(f)$ and $g(b) \notin \mathrm{Im}(f)$. Then $g(a) = f(z)$ for some $z \in X$. Note that by (3), this gives $a = e(z)$. For any $y \in X$ with $b \leq e(y)$ we have $g(b) \leq g(e(y)) = f(y)$, so $f(z) \leq w_1 \leq \cdots \leq w_n \leq g(b) \leq f(y)$. Since f is strict and $w_1, ...w_n, g(b) \notin \mathrm{Im}(f)$ we have $f(z) \leq f(y)$, hence $z \leq y$. This gives $a = e(z) \leq e(y)$ for all y with $b \leq e(y)$. Using that e is meet dense, we have $a \leq b$. The case when $g(a) \notin \mathrm{Im}(f)$ and $g(b) \in \mathrm{Im}(f)$ is similar.

Finally, suppose $g(a), g(b) \notin \mathrm{Im}(f)$. Then for any $x, y \in X$ with $e(x) \leq a$ and $b \leq e(y)$ we have $f(x) = g(e(x)) \leq g(a)$ and $g(b) \leq g(e(y)) = f(y)$. So

$$f(x) \leq g(a) \leq w_1 \leq \cdots \leq w_n \leq g(b) \leq f(y).$$

Since none of $g(a), g(b), w_1, ..., w_n$ belong to $\mathrm{Im}(f)$, by strictness $f(x) \leq f(y)$, so $x \leq y$, hence $e(x) \leq e(y)$. It follows from join and meet density that $a \leq b$. $\square$

Theorem 5.5. *Let $(E, e), (F, f)$ be pseudo MacNeille completions of a pseudo ordered set X. Then there exists a unique order isomorphism $g : E \to F$ with $f = g \circ e$.*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "e"'] & F \\
E \arrow[ur, dashed, "g"'] &
\end{tikzcd}$$

Proof. Let $g_1 : E \to F$ be defined by $g_1(c) = \bigvee\{f(x) : e(x) \leq c\}$ for every $c \in E$ and $g_2 : F \to E$ is given by $g_2(a) = \bigvee\{e(x) : f(x) \leq a\}$ for each $a \in F$. By Lemma 5.4, g_1 and g_2 are order embeddings with $g_1 \circ e = f$ and $g_2 \circ f = e$. We have that $f(X) \subseteq g_1(E)$ since any element of $f(X)$ is of the form $g_1(e(x))$ for some $x \in X$. Similarly $e(X) \subseteq g_2(F)$. Since f is meet dense, for any $p \in F$ we have $p = \bigwedge\{f(x) : p \leq f(x)\}$. Since every $f(x) = g_1(a)$ for some $a \in E$, namely for $a = e(x)$, we have $p = \bigwedge\{g_1(a) : p \leq g_1(a)\}$. So g_1 is meet dense. Similarly g_2 is meet dense. For any $a \in E$, $g_2(g_1(a)) = g_2(\bigvee\{f(x) : e(x) \leq a\})$. By Proposition 5.2, since g_2 is strict and meet dense, it preserves existing joins. Since g_2 preserves joins and $g_2 \circ f = e$, we have $g_2(g_1(a)) = \bigvee\{e(x) : e(x) \leq a\}$. Then as e is join dense, we have $g_2(g_1(a)) = a$. So $g_2 \circ g_1$ is the identity map on E, and similarly $g_1 \circ g_2$ is the identity on F. Therefore, g_1 is a bijection and an order embedding. Thus, g_1 is an order isomorphism. Now suppose that $h : E \to F$ is another order isomorphism such that $h \circ e = f$. Let $c \in E$. Since e is join dense, $c = \bigvee\{e(x) : e(x) \leq c\}$. Since g, h are order isomorphisms, they preserve joins. So $g(c) = \bigvee\{g(e(x)) : e(x) \leq c\} = \bigvee\{h(e(x)) : e(x) \leq c\} = h(c)$. $\square$

This shows that a pseudo ordered set has, up to unique commuting isomorphism, at most one pseudo MacNeille completion. We now turn to existence of a pseudo MacNeille completion. There are two approaches. One is a direct one, through modification of the construction of the MacNeille completion by normal ideals. We will only briefly mention this. We will focus on a more transparent method via the MacNeille completion of the covering poset.

Definition 5.6. For a pseudo ordered set X, let $(\mathrm{M}\,\Gamma(X), \iota)$ be the MacNeille completion of its covering poset. We assume that $\mathrm{M}\,\Gamma(X)$ is a complete lattice that contains $\Gamma(X)$ as a sublattice, that $\iota : \Gamma(X) \to \mathrm{M}\,\Gamma(X)$ is the inclusion map, and that ι is join and meet dense.

We begin with X pseudo ordered by $\leq$. Proposition 3.3 gives a partial ordering $\sqsubseteq$ on its covering poset $\Gamma(X)$. We also use $\sqsubseteq$ for the partial ordering on the MacNeille completion $\mathrm{M}\,\Gamma(X)$. Definition 4.1 specialized to this situation provides an equivalence relation θ on $\mathrm{M}\,\Gamma(X)$ where

$$x \,\theta\, y \quad \text{iff} \quad x = y \quad \text{or} \quad a^- \sqsubseteq x, y \sqsubseteq a^+ \text{ for some } a \in X$$

Proposition 4.2 gives that θ is a convex bounded pseudo congruence, and that the equivalence class x/θ is a singleton, or an interval $[a^-, a^+]$ for some $a \in X$, with values of $(x/\theta)^l$ and $(x/\theta)^u$ given accordingly. We use $\mathrm{M}(X)$ for the quotient $\mathrm{M}\,\Gamma(X)/\theta$. Corollary 4.3 gives that $\mathrm{M}(X)$ is a complete trellis under the pseudo ordering $\trianglelefteq$ where

$$x/\theta \;\trianglelefteq\; y/\theta \quad \text{iff} \quad (x/\theta)^l \sqsubseteq (y/\theta)^u.$$

Finally, Proposition 4.7 shows that the map $g : X \to \mathrm{M}(X)$ given by $g(x) = x^+/\theta$ is a strict order embedding. This completion $g : X \to \mathrm{M}(X)$ is the completion described in Section 4 using the MacNeille completion of the covering poset.

Theorem 5.7. *For a pseudo ordered set X, the completion $g : X \to \mathrm{M}(X)$ constructed using the MacNeille completion of the covering poset is a pseudo MacNeille completion of X.*

Proof. All that remains to be shown is that g is meet and join dense. We will show that it is meet dense, join density follows by symmetry. Let $x/\theta \in \mathrm{M}(X)$. We must show

$$x/\theta = \bigwedge\{g(c) : x/\theta \trianglelefteq g(c)\}.$$

Since $\mathrm{M}(X)$ is complete, there is $\gamma \in \mathrm{M}\,\Gamma(X)$ with $\gamma/\theta = \bigwedge\{g(c) : x/\theta \trianglelefteq g(c)\}$. It is clear that $x/\theta \trianglelefteq \gamma/\theta$, so we must prove that $\gamma/\theta \trianglelefteq x/\theta$. If x/θ is an interval $[a^-, a^+]$, then $x/\theta = g(a)$, and this is clear. It remains to consider the case when x/θ is the singleton $\{x\}$. For this, we consider separately the cases when γ/θ is an interval $[a^-, a^+]$, and when γ/θ is the singleton $\{\gamma\}$.

Assume γ/θ is the interval $[a^-, a^+]$ and x/θ is $\{x\}$. Having $\gamma/\theta \trianglelefteq x/\theta$ is equivalent to having $a^- \sqsubseteq x$. Since $\Gamma(X)$ is meet dense in its MacNeille completion, to show $a^- \sqsubseteq x$, it is sufficient to show that a^- lies beneath each element of $\Gamma(X)$ that lies above x. So we must show for each $c \in X$ that

$$x \sqsubseteq c^+ \Rightarrow a^- \sqsubseteq c^+ \quad \text{and} \quad x \sqsubseteq c^- \Rightarrow a^- \sqsubseteq c^-.$$

Suppose $x \sqsubseteq c^+$. This implies that $x/\theta \trianglelefteq g(c)$. Since γ/θ is the meet of all such elements, we have $\gamma/\theta \trianglelefteq g(c)$, and this implies $a^- \sqsubseteq c^+$. This provides the first item. For the second item, assume $x \sqsubseteq c^-$. By Proposition 3.3.4, to show that $a^- \sqsubseteq c^-$, we must show that if $u \in X$, then $c \leq u$ implies $a \leq u$. By Proposition 3.3.3, $c \leq u$ gives $c^- \sqsubseteq u^+$, so $x \sqsubseteq c^- \sqsubseteq u^+$. Since $\sqsubseteq$ is a partial ordering, $x \sqsubseteq u^+$, and this gives $x/\theta \trianglelefteq g(u)$. From the definition of γ/θ as a certain meet, we then have $\gamma/\theta \trianglelefteq g(u)$. This implies that $a^- \sqsubseteq u^+$. Then $a \leq u$ by Proposition 3.3.3.

It remains to show that $\gamma/\theta \trianglelefteq x/\theta$ in the case when γ/θ is the singleton $\{\gamma\}$. Since we are assuming that x/θ is the singleton $\{x\}$, this amounts to showing that $\gamma \sqsubseteq x$. Since $\Gamma(X)$ is meet dense in its MacNeille completion, it is enough to show

$$x \sqsubseteq c^+ \Rightarrow \gamma \sqsubseteq c^+ \quad \text{and} \quad x \sqsubseteq c^- \Rightarrow \gamma \sqsubseteq c^-.$$

If $x \sqsubseteq c^+$, then $x/\theta \trianglelefteq g(c)$. By the definition of γ/θ as a certain meet, we have $\gamma/\theta \trianglelefteq g(c)$, and this gives $\gamma \sqsubseteq c^+$. This establishes the first item. For the second item, suppose $x \sqsubseteq c^-$. To show that $\gamma \sqsubseteq c^-$, since $\Gamma(X)$ is join dense in its MacNeille completion, it is enough to show that each element of $\Gamma(X)$ that lies beneath γ also lies beneath c^-. So, under the assumption that $x \sqsubseteq c^-$ we must show

$$b^+ \sqsubseteq \gamma \Rightarrow b^+ \sqsubseteq c^- \quad \text{and} \quad b^- \sqsubseteq \gamma \Rightarrow b^- \sqsubseteq c^-.$$

Assume $b^+ \sqsubseteq \gamma$. To show $b^+ \sqsubseteq c^-$, by Proposition 3.3.2 we must show that $\ell \leq b$ and $c \leq u$ imply $\ell \leq u$. By Proposition 3.3, these conditions imply $\ell^- \sqsubseteq b^+ \sqsubseteq \gamma$ and $x \sqsubseteq c^- \sqsubseteq u^+$. Since $\sqsubseteq$ is a partial ordering, we have $x \sqsubseteq u^+$. This gives $x/\theta \trianglelefteq g(u)$, and by the definition of γ/θ as a meet, we have $\gamma/\theta \trianglelefteq g(u)$. This gives $\gamma \sqsubseteq u^+$. Since $\ell^- \sqsubseteq \gamma$, this gives $\ell^- \sqsubseteq u^+$, so by Proposition 3.3.3, $\ell \leq u$ as required. This establishes the first item. For the second item, suppose $b^- \sqsubseteq \gamma$. To show $b^- \sqsubseteq c^-$, by Proposition 3.3.4 we must show that $c \leq u$ implies $b \leq u$. By Proposition 3.3.3, $c \leq u$ gives $x \sqsubseteq c^- \sqsubseteq u^+$. Since $\sqsubseteq$ is transitive, $x \sqsubseteq u^+$, so $x/\theta \trianglelefteq g(u)$. The definition of γ/θ then gives $\gamma/\theta \trianglelefteq g(u)$, hence $b^- \sqsubseteq \gamma \sqsubseteq u^+$. So $b^- \sqsubseteq u^+$ and by Proposition 3.3.3 we have $b \leq u$ as required. □

Remark 5.8. There is an alternate approach to construct a pseudo MacNeille completion of a pseudo ordered set X that at first looks quite similar to the construction of the MacNeille completion of a poset via normal ideals. Say a subset $P \subseteq X$ is a normal ideal if $P = LU(P)$. Let M be the set of normal ideals of the form $LU(\{a\})$ for some $a \in X$, and let N be the set of all normal ideals of X that do not have a join. Note that there can be normal ideals of X not of either form, such as $L(\{a\})$ for an element a that is not transitive. Define a relation $\trianglelefteq$ on the set $M \cup N$ as follows. For $a, b \in X$ and $P, Q \in N$:

(1) $LU(\{a\}) \trianglelefteq LU(\{b\})$ iff $LU(\{a\}) \subseteq L(\{b\})$
(2) $LU(\{a\}) \trianglelefteq P$ iff $LU(\{a\}) \subseteq P$
(3) $P \trianglelefteq LU(\{a\})$ iff $P \subseteq L(\{a\})$
(4) $P \trianglelefteq Q$ iff $P \subseteq Q$

One can show directly that $M \cup N$ is a complete trellis, and that the map $e : X \to M \cup N$ defined by $e(a) = LU(\{a\})$ is a strict order embedding that is join and meet dense. For details, see [2]. However, there is a lack of transparency with this construction and why it should work. Indeed, it was by peering into the workings of this construction that we discovered the covering poset, bounded convex pseudo congruences, and the general method of completing a pseudo order via a completion of its partial order.

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