ZETA FUNCTIONS OF FINITE GRAPHS AND COVERINGS

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Abstract. The classical Riemann zeta function has important variants in a number of mathematical contexts. In this expository paper, we introduce the zeta function of finite graphs and study their interactions with the topological structure of covering spaces. We define the Ihara (vertex) zeta function $\zeta_X(u)$ for a finite connected graph X, and present a proof of its determinant formula. We show that if $Y \to X$ is a finite covering of graphs, then $\zeta_X(u)^{-1}$ divides $\zeta_Y(u)^{-1}$.

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1. Introduction

The classical Riemann zeta function $\zeta(s)$ is arguably one of the most celebrated functions in the history of mathematics, known even to non-mathematicians due largely to the notorious Riemann Hypothesis – a critically important unsolved mathematical problem having deep implications on the distribution of primes. Invented to address questions around prime distributions in analytic number theory, the notion of a zeta function has been generalized to various forms in different areas of mathematics and physics, playing important and sometimes central roles in these fields. For example, in arithmetic geometry the study of zeta functions of smooth projective varieties over finite fields turns out to be highly influential. In fact, the decades of efforts in proving the analogue of Riemann Hypothesis in this algebro-geometric setting (as a part of the Weil conjectures) has re-shaped modern algebraic geometry. Other notable zeta functions in other mathematical disciplines include the Dedekind zeta function defined for algebraic number fields, the Selberg zeta function in differential geometry, and the Artin-Mazur zeta function in dynamical systems.

The goal of our research is to explore yet another beautiful interaction of zeta functions, with the topological structure of covering spaces. More precisely, we make sense of the zeta function $\zeta_X(u)$ for any finite connected graph X, and explore how such graph-theoretic zeta functions interact with finite coverings of graphs.

Example 1.1. Below is a 2-sheeted covering of the tetrahedron graph.

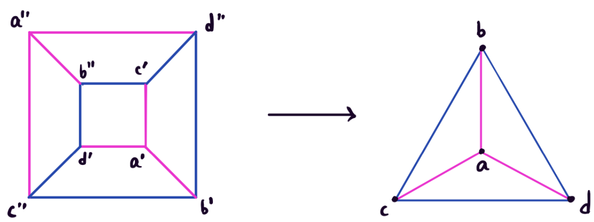

FIGURE 1. A 2-sheeted cover of the tetrahedron graph.

1.1. **The classical Riemann zeta function.** We begin with a brief historical account of the story. Perhaps one first encounters the classical Riemann zeta function in a standard calculus course. Indeed, the *Riemann zeta function*

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots = \sum_{n\geq 1} \frac{1}{n^s} \tag{1}$$

is a particular example of a Dirichlet series, and one learns that the series in Equation (1) converges whenever $s > 1$, and diverges otherwise. For example, when $s = 1$ one obtains the harmonic series which is one of the most familiar examples of divergent series.

At this point the zeta function $\zeta(s)$ is a single-variable, real-valued function defined for all real numbers $s > 1$. It is then a reasonable question to ask for precise values of $\zeta(s)$, especially when s is an integer greater than one. For example, the problem of determining $\zeta(2)$, known as the *Basel problem*, was first proposed by Pietro Mengoli in 1650. Progress in finding precise zeta values are pioneered by the ground-breaking work of Leonhard Euler, who completely solved the Basel problem:

$$\sum_{n\geq 1} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6}. \tag{2}$$

The most surprising feature of Equation (2) is the appearance of (the square of) π in the formula. There has been a number of proofs of Equation (2) since the original one of Euler, and we present an elementary proof in the Appendix where only techniques from integral calculus are involved.

Euler achieved much more than merely solving the Basel problem. In fact, in 1735 he obtained a closed-form description of $\zeta(s)$ for all positive even integers s:

$$\zeta(2n) = (-1)^{n-1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}, \forall n \geq 1.$$

Here the numbers B_{2n} are *Bernoulli numbers* characterized by the generating function

$$\frac{x}{e^x - 1} = 1 + \sum_{k\geq 1} B_k \frac{x^k}{k!}.$$

In contrast, precise zeta values at odd integers are much harder to determine. To this day, it is not even known whether $\zeta(3)$ is transcendental.

One remarkable contribution of Euler in this subject is to relate zeta function with primes. Recall that a positive integer greater than one is a *prime number* (or just a *prime* for short), if its only factors are one and itself. Prime numbers are basic building blocks of integers (via the unique factorization) and human beings have a long history of investigating both properties of primes and the distribution among integers. The proof of Euclid from 300 B.C. that there exists infinitely many primes is acknowledged as one of the most beautiful proofs in the history of mathematics. In 1737, Euler observed that $\zeta(s)$ can be expressed purely in terms of primes:

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} (1-p^{-s})^{-1}. \tag{3}$$

The intuition of Equation (3) is coming from the sieve of Eratosthenes – another Greek mathematician around Euclid's time.

The next dominant figure in the history of this subject is Bernhard Riemann, who approached $\zeta(s)$ from complex analysis, with the motivation of studying the distribution of primes. As a function of complex variable, $\zeta(s)$ has $\mathrm{Re}(s) > 1$ as its domain of convergence. In his famous 1859 paper, he pointed out that $\zeta(s)$ can be analytically continued to a meromorphic function over the entire complex plane, with the single simple pole at $s = 1$. This extended $\zeta(s)$ has lots of trivial zeros, e.g., one at every negative even integer, but there are also non-trivial ones. In the same paper, Riemann conjectured that all non-trivial zeros of $\zeta(s)$ should lie on the line $\mathrm{Re}(s) = \frac{1}{2}$, and expected that a positive answer to the conjecture would help to prove the *Prime Number Theorem*. For a real number $x \geq 2$, write $\pi(x)$ for the number of primes less than or equal to x. The function $\pi(x)$ is known as the prime-counting function, and studying the distribution of primes is essentially amounts to understanding the asymptotic behavior of $\pi(x)$ as x tends to infinity. The Prime Number Theorem states that

$$\pi(x) \sim \frac{x}{\ln x} \sim \mathrm{Li}(x), \quad x \to \infty,$$

where $\mathrm{Li}(x) = \int_2^x \frac{1}{\ln t} dt$ is the logarithmic integration function. With the help of Euler's Equation (3) and various other techniques, Riemann was able to relate $\pi(x)$ to the zeros of $\zeta(s)$ and made the conjecture that all non-trivial zeros of $\zeta(s)$ should lie on the line $\mathrm{Re}(s) = \frac{1}{2}$. This *Riemann Hypothesis* has not been proved or disproved as of the year of 2025, though the Prime Number Theorem itself was proved already in 1896, independently by Hadamard and de la Vallée Poussin.

1.2. **Main results.** The main focus of our paper is to develop a theory of zeta functions in a combinatorial and topological context, namely that of finite graphs and their coverings. As one-dimensional CW complexes, graphs are among the most fundamental examples of topological spaces, and coverings of graphs give rise to a rich source of examples of covering spaces. However, difficulties immediately arise as one attempts to define the zeta function for a graph as an infinite series, as there is no obvious way to encode the basic features of the given graph (e.g., its data of edges and vertices) in the formula. At this point, Euler's formula Equation (3) provides a key insight – if one is able to make sense of "primes" in a graph, then a reasonable definition would be to take the zeta function as appropriate products over all primes. This is exactly how the story is developed in history. Consider a finite connected graph X without degree-one vertices. One can make sense of prime loops in X – roughly speaking, they are backtrackless, tailless, primitive paths in X (see Section 3 for details). The zeta function of X

is then defined as

$$\zeta_X(u) = \prod_{[P]} (1 - u^{\nu(P)})^{-1},$$

where the product is taken over equivalence classes of primes in X, and $\nu(P)$ denotes the length of the prime P (see Definition 3.3 for details).

Example 1.2. Zeta functions of cyclic graphs are straightforward to compute, since they contain exactly two prime loops up to equivalence.

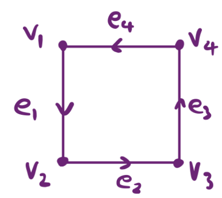

FIGURE 2. A cyclic graph C_4 with zeta function $\zeta_{C_4}(u) = (1 - u^4)^{-2}$.

This notion of zeta function of graphs was first introduced by Ihara [Iha66] for the purpose of studying discrete subgroups of projective linear groups over p-adic fields, and was further developed by Hashimoto [Has89] and Bass [Bas92]. In particular, Hashimoto and Bass independently proved an equivalent form of the graph-theoretic zeta function, known as the determinant formula, and this is the first main result we would like to present:

Theorem 1.3. *The inverse of the graph zeta function $\zeta_X(u)$ equals*

$$(1 - u^2)^{r_X - 1} \det(I - A_X u + Q_X u^2).$$

In the above theorem, A_X is the adjacency matrix of X, and Q_X is a diagonal matrix whose entries along the main diagonal come from the degree of vertices in X. The number r_X is the rank of the fundamental group of X, so that $r_X - 1$ equals the number of edges in X minus the number of vertices. (See Section 3 for detailed explanations.)

Example 1.4. In Example 1.1, the tetrahedron graph T has adjacency matrix

$$A_T = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

and therefore

$$\begin{aligned} \zeta_T(u)^{-1} &= (1 - u^2)^{r_T - 1} \det(I - A_T u + Q_T u^2) \\ &= (1 - u^2)^2 \det \begin{bmatrix} 1 + 2u^2 & -u & -u & -u \\ -u & 1 + 2u^2 & -u & -u \\ -u & -u & 1 + 2u^2 & -u \\ -u & -u & -u & 1 + 2u^2 \end{bmatrix} \\ &= (1 - u^2)^2 (1 - u)(1 - 2u)(1 + u + 2u^2)^3. \end{aligned}$$

The interaction of graph zeta functions with the topological structure of covering spaces was extensively studied by Stark and Terras in a sequence of papers [ST96, ST00, ST07]. In particular, it is proved that if $Y \to X$ is a finite covering of graphs then the inverse of the zeta functions satisfy a divisibility property. This is the second main result we would like to present in this paper:

Theorem 1.5. Let $p : Y \to X$ be a finite covering of graphs. Then $\zeta_X(u)^{-1}$ divides $\zeta_Y(u)^{-1}$.

Example 1.6. In Example 1.1, the covering graph $\tilde{T}$ (namely the cube graph) has

$$\zeta_{\tilde{T}}(u)^{-1} = (1-u^2)^4(1-4u^2)(1-u+2u^2)^3(1+u+2u^2)^3,$$

which is clearly divisible by $\zeta_T(u)^{-1} = (1-u^2)^2(1-u)(1-2u)(1+u+2u^2)^3$.

1.3. **Structure of the paper.** Later sections of the paper are organized as follows. In Section 2, we survey the needed background knowledge from algebraic topology, mainly the material of fundamental groups and covering spaces, with [Hat02] as our main reference. Next in section Section 3 we specialize covering spaces to the case of finite coverings of graphs. We set up the basic notions about finite graphs, and define the graph-theoretic zeta function. The final part of the paper, Section 4, is dedicated to proving the main results, Theorem 1.3 and Theorem 1.5. Our proofs follow largely from the excellent book of Terras [Ter10]. As promised earlier, we give an elementary solution of the Basel problem in the Appendix.

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2. Background: Fundamental groups and covering spaces

2.1. **The notion of a homotopy.** In topology, we often care about shapes up to "continuous deformation." A *homeomorphism* is a bijective, continuous map with a continuous inverse—it allows us to transform one space to another by bending, stretching, and twisting, but without tearing things apart or gluing them together. It is a very strong equivalence, implying spaces are topologically "identical," meaning all topological properties are preserved [Sti16, Definition 1.10].

However, proving or disproving that two spaces are homeomorphic can be difficult. To study spaces more flexibly, we introduce a weaker equivalence relation: *homotopy*. Homotopy gives us a more relaxed way of classifying spaces (and maps between them) by allowing further deformation through *shrinking* and *expanding*, but still no gluing or tearing [Hu23]. This is what helps classify spaces based on properties like "holes" or "connectivity."

Example 2.1. A classic illustrative example is the deformation of a coffee mug into a donut: both possess a single "hole," and while they may appear different geometrically, they are in fact homeomorphic (and thus homotopy equivalent) because they can be deformed into each other without cutting or gluing.

Example 2.2. Consider the digits '0', '6', and '9' as one-dimensional figures drawn in a plane. All three can be deformed into a simple circle S^1. While '0' is directly homeomorphic to S^1, '6' and '9' are homeomorphic to eachother but not S^1 (as their "tails" can't be smoothly absorbed). However, these spaces are all considered *homotopy equivalent* to S^1. This illustrates how homotopy helps categorize shapes by their essential features, such as the presence of a single loop.

Definition 2.3 (Homotopy)**.** Let X and Y be topological spaces. Two continuous maps $f, g : X \to Y$ are said to be *homotopic*, if there exists a continuous map

$$H : X \times \mathrm{I} \to Y$$

such that

$$H(x, 0) = f(x) \quad \text{and} \quad H(x, 1) = g(x) \quad \text{for all } x \in X.$$

In this case, we say that f is homotopic to g, and such an H is called a *homotopy* from f to g, and we write $f \simeq g$.

Here, the unit interval $\mathrm{I} = [0, 1]$ plays the role of a "time" parameter. At time $t = 0$, we start with the function f, and by $t = 1$, we continuously deform it into the function g. Each fixed time $t \in \mathrm{I}$ gives a map

$$H_t(x) := H(x, t),$$

so the homotopy H can be viewed as a continuous family of maps $\{H_t : X \to Y\}_{t \in \mathrm{I}}$ interpolating between f and g.

Intuitively, f is homotopic to g if f can be continuously deformed to g inside Y within unit time. Geometrically, we can think of $X \times \mathrm{I}$ as a cylinder. The homotopy H maps this cylinder into Y, where the "bottom" $X \times \{0\}$ is mapped by f, and the "top" $X \times \{1\}$ is mapped by g. Each point $x \in X$ traces a continuous path $t \mapsto H(x, t)$ in Y, showing how $f(x)$ transitions into $g(x)$.

More than just a concept, homotopy defines an equivalence relation on continuous maps.

Proposition 2.4. Homotopy is an equivalence relation on the set $C(X, Y)$ of continuous maps from X to Y.

Sketch of Proof. We must check that $\simeq$ is reflexive, symmetric, and transitive [Hu23, Page 13]:

- **Reflexivity:** $f \simeq f$ via the constant homotopy $H(x, t) = f(x)$.
- **Symmetry:** If $f \simeq g$ via H, then $g \simeq f$ via $\tilde{H}(x, t) := H(x, 1 - t)$. The continuity of $\tilde{H}$ follows from the continuity of H.
- **Transitivity:** If $f \simeq g$ via H_1 and $g \simeq h$ via H_2, define:

$$H(x, t) = \begin{cases} H_1(x, 2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ H_2(x, 2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

  The map H is continuous. Its continuity at $t = 1/2$ follows from $H_1(x, 1) = g(x)$ and $H_2(x, 0) = g(x)$. This map H provides a homotopy from f to h.

□

Because homotopy is an equivalence relation, it partitions $C(X, Y)$ into *homotopy classes*. The homotopy class of a map f is denoted by $[f]$, and the set of all such classes is denoted $[X, Y]$ [Hu23, Page 13].

We now introduce some important concepts related to homotopy:

Definition 2.5 (Null-Homotopic Map)**.** A map $f : X \to Y$ is called *null-homotopic* if it is homotopic to a constant map.

Intuitively, a null-homotopic map can be 'shrunk' continuously to a point.

Example 2.6. Any map from any space X into a convex subset of $\mathbb{R}^n$ (such as an open disk D^n or $\mathbb{R}^n$ itself) is null-homotopic, as it can be linearly contracted to a point.

Definition 2.7 (Contractible Space)**.** A space X is said to be *contractible* if its identity map $\mathrm{id}_X : X \to X$ is null-homotopic. Equivalently, a space is contractible if it is homotopy equivalent to a single point $X \simeq \{*\}$.

Contractible spaces are important because they behave like a point in homotopy theory, and their fundamental group is always trivial.

Example 2.8. Examples of contractible spaces include any convex subset of $\mathbb{R}^n$, such as an open disk D^n or the entire space $\mathbb{R}^n$ itself. Conversely, the unit circle S^1 is not contractible, which will be demonstrated by its non-trivial fundamental group.

Definition 2.9 (Homotopy Equivalence)**.** A continuous map $f : X \to Y$ is called a *homotopy equivalence* if there exists a continuous map $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. In this case, we say that X and Y are *homotopy equivalent*, denoted $X \simeq Y$, and that they share the same *homotopy type*.

Example 2.10. The real plane $\mathbb{R}^2$ is homotopy equivalent to a single point $\{*\}$ [Sti16, Example 1.15]. This signifies that, from a homotopy perspective, $\mathbb{R}^2$ can be continuously shrunk to a point.

Example 2.11. The punctured plane $\mathbb{R}^2 \setminus \{0\}$ is homotopy equivalent to the unit circle S^1. This is a key example, as it shows that the "hole" at the origin is the essential feature preserving the homotopy type with S^1, despite the large difference in overall shape.

2.1.1. *Important Special Cases and Related Notions.*

Definition 2.12 (Path Homotopy)**.** A *path homotopy* is a special case of homotopy between paths $\gamma, \gamma' : \mathrm{I} \to X$ where the endpoints remain fixed throughout the deformation. A path homotopy $H : \mathrm{I} \times \mathrm{I} \to X$ satisfies:

$$H(0,t) = \gamma(0) = \gamma'(0), \quad \text{and} \quad H(1,t) = \gamma(1) = \gamma'(1), \quad \text{for all } t \in \mathrm{I}.$$

We denote this by $\gamma \simeq_{\mathrm{rel}\ \partial\mathrm{I}} \gamma'$ or simply $\gamma \simeq \gamma'$ when the context of fixed endpoints is clear, where $\partial\mathrm{I} = \{0,1\}$ is the boundary of the unit interval. This concept is foundational for defining the *fundamental group*.

Let $\gamma : \mathrm{I} \to X$ be a path from x to y, and $\sigma : \mathrm{I} \to X$ be a path from y to z. We can *concatenate* these two paths to form a new path $\gamma \cdot \sigma : \mathrm{I} \to X$ from x to z, defined as:

$$(\gamma \cdot \sigma)(t) = \begin{cases} \gamma(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ \sigma(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

This operation is called path concatenation or path multiplication. Intuitively, this definition traverses the path γ in the first half of the time interval I, and then the path σ in the second half.

The inverse path of γ, denoted γ^{-1}, is defined as $\gamma^{-1}(t) = \gamma(1-t)$ for $t \in \mathrm{I}$. This path traverses γ in the reverse direction.

Path concatenation has several important properties under path homotopy:

- **Associativity (up to homotopy):** Given paths $\alpha : x \to y$, $\beta : y \to z$, and $\gamma : z \to w$, we have $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha \cdot (\beta \cdot \gamma)$.
- **Identity (up to homotopy):** For a path $\alpha : x \to y$, let c_x be the constant path at x (i.e., $c_x(t) = x$ for all $t \in \mathrm{I}$). Then $\alpha \cdot c_y \simeq \alpha$ and $c_x \cdot \alpha \simeq \alpha$.
- **Inverse (up to homotopy):** For a path $\alpha : x \to y$, we have $\alpha \cdot \alpha^{-1} \simeq c_x$ (a loop at x) and $\alpha^{-1} \cdot \alpha \simeq c_y$ (a loop at y).

These properties together will allow us to define the product of loops in the fundamental group.

Definition 2.13 (Deformation Retraction)**.** A map $r : X \to A \subseteq X$ is a *deformation retraction* if r is a retraction (i.e., $r|_A = \mathrm{id}_A$) and the inclusion map $i : A \hookrightarrow X$ is a homotopy equivalence such that $i \circ r \simeq \mathrm{id}_X$. In this case, X can be continuously "shrunk" onto A while preserving homotopy type.

Homotopy provides a robust framework for understanding spaces up to deformation, helping us classify them based on qualitative features like the number of holes, paths, and loops. This is incredibly useful because the *fundamental group* is constructed by considering homotopy classes of *loops* (paths whose start and end points coincide) based at a fixed point. The crucial condition of path homotopy, which requires endpoints to remain fixed, ensures that the concatenation of loops is well-defined as a group operation on these homotopy classes. This leads directly to the definition of the fundamental group, which is the next major topic in our development.

2.2. **The fundamental group of a space.** We have already seen the appearance of a notion of *fundamental group* in the determinant formula for the graph zeta function Theorem 1.3 where r_x was defined as the rank of the fundamental group. Its presence is no surprise since, intuitively, the fundamental group is a collection of all possible loops in a space, up to homotopy, and for which an extra group structure emerges. In fact, the fundamental group is a topological invarient which is a foundational concept in algebraic-topology. It is a vital concept for the connection between topology and algebra, where it was realized that covering spaces of topological spaces correspond to subgroups of the fundamental group of that space. This bridges algebra and topology, allowing us to study problems in algebra from a topological perspective and vice versa. A classical result of this is in a topological proof that the subgroups of a free group are also free. Now, In order to form a definition of the fundamental group we must start with *loops*.

Definition 2.14. (Loop) Let X be a topological space. A path σ which both starts and ends at the point x_0 in X, i.e. $\sigma(0) = \sigma(1) = x_0$, is called a *loop* in X *based* at x_0.

Since there is an equivalence relation under path homotopy, we may create the notion of the set of all homotopy classes of loops in X based at x_0. We shall denote this set as $\pi_1(X, x_0)$. A typical element of $\pi_1(X, x_0)$ would be like $[\sigma]$ where σ is a loop based at x_0 and is a representative of the homotopy class of loops at x_0 to which σ belongs. This is to say that, $[\sigma] = [\tau]$ precisely when σ is path homotopic to τ, as loops.

Now, two loops at x_0 may be concatenated to create a third loop at x_0. Treating loop concatenation as a product of loops, does this form a group structure on $\pi_1(X, x_0)$? Indeed, the answer is yes.

Claim 2.15. $\pi_1(X, x_0)$ becomes a group under path concatenation.

Proof. First,

$$\pi_1(X, x_0) \times \pi_1(X, x_0) \to \pi_1(X, x_0)$$
$$([\gamma], [\sigma]) \mapsto [\gamma \cdot \sigma]$$

is well defined. Take any two paths as representatives of $[\gamma]$, say γ and γ', likewise for σ and σ'. Since path concatenation is well-defined up to homotopy and $\gamma \simeq \gamma', \sigma \simeq \sigma'$, then $\gamma \cdot \sigma \simeq \gamma' \cdot \sigma'$, thus we may pick either product as the representative of $[\gamma \cdot \sigma]$ without losing generality.

In a similar manner, associativity $([\gamma][\sigma])[\tau] = [\gamma]([\sigma][\tau])$ follows from the associativity of path concatentation up to homotopy, the identity is $[c_{x_0}]$, and it is clear that $[\sigma]^{-1}$ is simply $[\sigma^{-1}]$. □

Now we are prepared to define the notion of the *fundamental group*.

Definition 2.16. Let X be a topological space and choose some base point $x_0 \in X$. The set $\pi_1(X, x_0)$ together with the operation of path concatentation is called the *fundamental group* of X based at x_0. Henceforth, $\pi_1(X, x_0)$ will refer to the *fundamental group* itself.

Notice that the above definition for Definition 2.16 depends on a choice of base point x_0, but it turns out that, up to isomorphism, the choice of base point is irrelevant. In fact, this brings us to an important property.

Theorem 2.17. If a topological space X is path connected, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for every $x_0, x_1 \in X$. We may refer to this isomorphism class simply as the fundamental group of X and denote it as $\pi_1(X)$.

Proof. The main idea is to construct a bijection and verify that it satisfies homomorphism. Consider the map,

$$f : \pi_1(X, x_0) \to \pi_1(X, x_1)$$
$$[\sigma] \mapsto [\gamma^{-1}\sigma\gamma]$$

where γ is a path from x_0 to x_1. f is well defined in the same manner as the proof of Claim 2.15. Also,

$$f([\sigma][\tau]) = [\gamma^{-1}\sigma\tau\gamma] = [\gamma^{-1}\sigma\gamma\gamma^{-1}\tau\gamma] = f([\sigma])f([\tau])$$

So, f satisfies homomorphism. Next, we need only notice that $f^{-1}(\sigma) = [\gamma\sigma\gamma^{-1}]$ works to bring us to an isomorphism. □

In essence, this means there is a group for every space and we can discuss the fundamental group of a space without the necessity of specifying a base point.

Looking forward to the a treatment of covering spaces a useful fact is that the fundamental group is functorial.

Theorem 2.18. Suppose that $f : (X, x_0) \to (Y, y_0)$ is a continuous based map between topological spaces. Then, f induces a group homomorphism.

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$
$$[\sigma] \mapsto [f \circ \ \sigma]$$

Proof. The homomorphism f_* is well defined since from the definition of loop and $f \cdot \sigma$ is a mapping such that $I \times I \to X \to Y$. Clearly the image is again a loop in $\pi_1(Y, y_0)$. Also, that f_* satisfies homomorphism follows directly from applying the definition of path concatentation to $f \circ (\sigma \cdot \tau)$ $(f \circ \sigma) \cdot (f \circ \tau)$. □

In fact, if f is an homotopy equivalence we can immediately say more, that f_* is an isomorphism.

Corollary 2.19. Suppose that $f : X \to Y$ is a continuous map between topological spaces and that f is a homotopy equivalence. Then, $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)$ is an isomorphism.

Thus, π_1 is homotopy invariant. We may deform a topological space in any way up to homotopy equivalence, say mush a donut into a coffee mug or even squish a 4 into a 0, and rest easy that this does not affect π_1. A straightforward demonstration of the significance of this quality is in the classical example that $\pi_1(S^1) \cong \mathbb{Z}$.

Example 2.20. The fundamental group of the cirvle is isomorphic to $\mathbb{Z}$.

Intuitively, we can imagine looping around the unit circle and counting how many times we pass the start point. Go around clockwise once, twice, three times and the positive integers come naturally, then we can go around the counterclockwise and say these loops correspond to negative integers.

More precisely, we can define a *winding function.*

$$W : \pi_1(S^1) \to \mathbb{Z}$$
$$[\sigma] \mapsto \tilde{\sigma}(1)$$

where $\tilde{\sigma}(s)$ is the *lift* from definition 2.26 of

$$p : \mathbb{R} \to S^1$$
$$s \mapsto e^{2\pi i s}$$

One can then show straightforwardly that W is well-defined, a group homomorphism, and is surjective and injective.

2.3. **Covering spaces.** The notion of the covering space will be introduced, to motivate the covering spaces of graphs occurring in the next seciton. Throughout the rest of this section, all spaces are assumed to be topological, and all maps are assumed to be continuous.

Definition 2.21. Let X be a space. A *covering space* over X is another space $\tilde{X}$, together with a map $p : \tilde{X} \to X$, if and only if for every element $x \in X$ there exists an open neighborhood $U_x \subset X$ of x, such that its preimage $p^{-1}(U_x) = \bigsqcup_\alpha V_\alpha$ is a union of disjoint open subsets $V_\alpha \subset \tilde{X}$, and p maps each V_α homeomorphically onto U_x. In this situation, $\tilde{X}$ is said to be the *total space*, and X is said to be the *base space*. (See [Hat02].)

Intuitively speaking, a covering space "covers down", so to speak, each U_x with a collection V_α. Also note that any space can act as its own covering space.

Example 2.22. Let

$$p : \mathbb{R} \to S^1$$
$$x \mapsto e^{2\pi i x}$$

be a map between the real line $\mathbb{R}$ and the unit circle S^1 in $\mathbb{C}$. It is easy to see that $\mathbb{R}$ is a covering space for S^1.

Example 2.23. Let $n \in \mathbb{Z}^+$ be an arbitary desired winding number, and let

$$p : W^1 \to S^1$$
$$z \mapsto z^n$$

be a map between two identical unit circles in the complex plane (i.e., we treat W^1 as a seperate identical copy of S^1, labeled differently to distinguish domain and range). When W^1 is homeomorphically deformed to a torus-shaped spring with n cycles, it becomes clear that this presentation of W^1 is a covering space for S^1.

Definition 2.24. Let $p : \tilde{X} \to X$ be a map from a covering space $\tilde{X}$ to its base space X, and let $x_0 \in X$ be an element. The preimage $p^{-1}(x_0)$ is said to be the *fiber* of x_0. If the cardinality of the fiber $p^{-1}(x_0) = n$ is finite, we say that $\tilde{X}$ is an *n-sheeted cover*, or an *n-fold cover*. (See [Hat02].)

The term "fiber" comes from the visualization of a bundle of fibers extending out of $p^{-1}(x_0)$ and closing onto x_0.

Example 2.25. In Example 2.22, we see that the fiber of $z_0 = 1$ in the unit circle S^1 is $\mathbb{Z}$ in $\mathbb{R}$. Similarly, in Example 2.23, the cardinality-n fiber of z_0 in S^1 is the set of nth-roots of unity in W^1. Note that W^1 is an n-fold cover.

Now we are going to examine some lifting lemmas. These will aid significantly in motivating our discussion of the action of the fundamental group and of the Galois group of a covering space. Let $p : \tilde{X} \to X$ map the covering space $\tilde{X}$ to its base space X, and let $x_0 \in X$ be an element. Additionally, let $\tilde{x}_0 \in p^{-1}(x_0)$ be an element of the fiber $p^{-1}(x_0)$. First, recall the following:

Definition 2.26. Let $f : Y \to X$ be a map between two spaces. Then a *lift* of f is a map $\tilde{f} : Y \to \tilde{X}$ such that $p \circ \tilde{f} = f$.

Lemma 2.27. Let Y be a connected space, with $f : Y \to X$ being a map. If $\tilde{f}_1, \tilde{f}_2$ are two lifts of f such that $\tilde{f}_1(y_0) = \tilde{f}_2(y_0)$ for some $y_0 \in Y$, then $\tilde{f}_1 = \tilde{f}_2$.

Proof. See Proposition 1.34 in [Hat02]. □

Lemma 2.28. Let f_0, f_1 be two maps from spaces Y to X, which are homotopic via $H : Y \times I \to X$. Additionally, let $\tilde{f}_0 : Y \to \tilde{X}$ be a lift of f_0. Then there exists a unique homotopy $\tilde{H} : Y \times I \to \tilde{X}$ which starts from $\tilde{f}_0$ and lifts the given homotopy H.

Proof. See Proposition 1.30 in [Hat02]. □

Lemma 2.29. Let γ be a path in X starting at x_0. Given any element $\tilde{x}_0$ in the fiber $p^{-1}(x_0)$, there exists a unique path $\tilde{\gamma}$ in $\tilde{X}$ which starts from $\tilde{x}_0$ and lifts γ.

Proof. In Lemma 2.28, let Y be a point. The result immediately follows. □

Lemma 2.30. Suppose that γ, σ are path homotopic paths in X from $x_0 \in X$ to $x_1 \in X$. If $\tilde{\gamma}, \tilde{\sigma}$ are lifts of γ, σ respectively sharing the same start point, then they are path homotopic, and in particular have the same endpoint.

Proof. Apply Lemmas 2.29 and 2.30. □

Now we are going to examine the action of the fundamental group on the fiber. It may be helpful to recall some of the following group-theoretic definitions (see [DF04] and [Armstrong]):

Definition 2.31. Let G be a group acting on the nonempty set A.

(1) The equivalence class $\{g \cdot a \mid g \in G\}$ is called the *orbit* of G containing a, notated as $G(a)$.
(2) The action of G on A is called *transitive* if and only if there is only one orbit, i.e., given any two elements $a, b \in A$ there is some $g \in G$ such that $a = g \cdot b$.
(3) The *stabilizer* of $s \in A$ in G is the set $G_a = \{g \in G \mid g \cdot s = s\}$.
(4) When the stabilizer for very $s \in A$ is the trivial subgroup e in G, then we say that G is a *free action* on A.

Proposition 2.32. The following action of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$ in $\tilde{X}$ is well-defined:

$$\pi_1(X, x_0) \times p^{-1} \to p^{-1}(x_0)$$
$$([\sigma_{x_0}],\, x_0) \mapsto \tilde{\sigma}_{\tilde{x}_0}(1)$$

where we adopt the following notational conventions: σ_{x_0} is a loop in X both starting and ending at x_0; the equivalence class $[\sigma_{x_0}] \in \pi_1(X, x_0)$ is the set of loops homotopic to σ_{x_0} in X; and $\tilde{\sigma}_{\tilde{x}_0}(1) \in p^{-1}(x_0)$ denotes the endpoint of the lifted path $\tilde{\sigma}_{\tilde{x}_0}$, which path $\tilde{\sigma}_{\tilde{x}_0}$ starts at $\tilde{x}_0$ and is lifted from the loop σ_{x_0} in X.

Proof. The uniqueness of path lifting follows from Lemmas 2.29 and 2.30. □

Note that this Proposition 2.32 serves more to define the group action of the fundamental group $\pi_1(X, x_0)$.

Example 2.33. In Example 2.23, let $x_0 = 1$ in S^1 be our basepoint, and let

$$p^{-1}(x_0) = \{e^{0\frac{2\pi i}{n}}, e^{1\frac{2\pi i}{n}}, \ldots, e^{(n-1)\frac{2\pi i}{n}}\}$$

in W^1 be its fiber. Let $\sigma^w_{x_0}$ be a loop in X traversing x_0 in w-cycles in the positive-w equals counterclockwise sense. Then, letting $\pi_1(S^1, x_0)$ act on $p^{-1}(x_0)$, we obtain

$$\tilde{\sigma}_{\tilde{x}_0}(1) = \begin{cases} e^{0\frac{2\pi i}{n}}\tilde{x}_0, w \in n\mathbb{Z} \\ e^{1\frac{2\pi i}{n}}\tilde{x}_0, w \in n\mathbb{Z}+1 \\ \ldots \\ e^{(n-1)\frac{2\pi i}{n}}\tilde{x}_0, w \in n\mathbb{Z}+(n-1) \end{cases}$$

where $n\mathbb{Z}+m$ denotes cosets of $\mathbb{Z}$. Note that this group action is isomorphic to

$$\begin{aligned} \mathbb{Z} \times \mathbb{Z}/_n &\to \mathbb{Z}/_n \\ (k, m) &\mapsto m + k \ (mod\, n) \end{aligned}$$

where $\mathbb{Z}/_n$ denotes the set of integers modulo-n.

Proposition 2.34. For any $\tilde{x}_0 \in p^{-1}(x_0)$, the group homomorphism $P_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective.

Proof. Let $\sigma, \gamma \in [\sigma]$ be any two path homotopic loops that fall in the same equivalence class $[\sigma]$, which class is in the image of P_*. Then, by Lemmas 2.29 and 2.30, their path lifts $\tilde{\sigma}, \tilde{\gamma}$ are also path homotopic, implying that they belong to the same equivalence class $[\tilde{\sigma}]$. Thus, $[\sigma]$ has only $[\tilde{\sigma}]$ as its preimage. □

Similar to Proposition 2.32, we'll use this Proposition 2.34 more as definition for P_*.

Example 2.35. From Examples 2.23 and 2.33, we can define P_* as

$$\begin{gathered} P_* : \pi_1(W^1) \to \pi_1(S^1) \\ [\tilde{\sigma}^w_{\tilde{x}_0}] \mapsto [\sigma^{nw}_{x_0}] \end{gathered}$$

where the basepoints of each fundamental group can be chosen arbitrarily for such a path-connected space (as shown in the previous subsection). Note that the loop class $[\tilde{\sigma}^w_{\tilde{x}_0}]$ in $\pi_1(W^1)$ gets mapped to the loop class $[\sigma^{nw}_{x_0}]$ in $\pi_1(S^1)$, with the latter having n-times the number of cycles as its preimage.

Propositions 2.36 through 2.40, in addition to Corollary 2.44 and Proposition 2.45, are rather useful facts concerning the Galois theory of covering spaces.

Proposition 2.36. The path $\tilde{\sigma}_{\tilde{x}_0}$ is a loop, if and only if $[\sigma] \in \pi_1(X, x_0)$ is in the image of $P_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$.

Proof. We prove the forward and backward implications:

($\Rightarrow$) Loops must map to loops.

($\Leftarrow$) Suppose $[\sigma] = P_*[\tilde{\tau}]$ for some $[\tau] \in \pi_1(\tilde{X}, \tilde{x}_0)$. Then both $[\tilde{\tau}]$ and $\tilde{\sigma}_{\tilde{x}_0}$ are lifts of σ starting at $\tilde{x}_0$. So $\tilde{\tau} = \tilde{\sigma}_{\tilde{x}_0}$ by Lemmas 2.29 and 2.30. □

Also, from now on, let's assume that the base space in a covering is path-connected.

Proposition 2.37. Suppose that $\tilde{X}$ is also path-connected. For every $y_0, y_1 \in p^{-1}(x_0)$, the groups $P_*\pi_1(\tilde{X}, y_0)$ and $P_*\pi_1(\tilde{X}, y_1)$ are conjugate subgroups in $\pi_1(X, x_0)$.

Proof. Let γ be a path from y_0 to y_1 in $\tilde{X}$. Then $P_*\pi_1(\tilde{X}, y_1) = [p \cdot \gamma]^{-1} P_*\pi_1(\tilde{X}, y_0)[p \cdot \gamma]$. □

Proposition 2.38. The covering space $\tilde{X}$ is path connected, if and only if $\pi(X, x_0)$ acts transitively on $p^{-1}(x_0)$.

Proof. We prove the forward and backward implications:

($\Rightarrow$) Suppose $\tilde{X}$ is path connected, and that $y_0, y_1 \in p^{-1}(x_0)$. Let γ be a path in $\tilde{X}$ from y_0 to y_1. Then $p \cdot \gamma$ is a loop in X at x_0, and $(p \tilde{\cdot} \gamma)_{y_0}(1) = \tilde{\gamma}(1) = y_1$. So $y_0 \cdot [p \cdot \gamma] = y_1$.

($\Leftarrow$) Suppose that the action is transitive, and $y, y' \in \tilde{X}$. Write x, x' for $p(y), p(y')$, respectively. Since X is path connected, there exists a path δ from x to x'. Now $\tilde{\delta}_y$ is a path in $\tilde{X}$ with start point y and endpoint $\tilde{\delta}_y(1) = y'' \in p^{-1}(x')$. Let $[\sigma] \in \pi_1(X, x')$ be such that $y'' \cdot [\sigma] = y'$. (That is, it has start point of y'' and enpoint of y'.) Now $\tilde{\delta}_y \cdot \tilde{\sigma}_{y''}$ is a path connecting y and y'. ☐

Proposition 2.39. The stabilizer of $y_0 \in p^{-1}(x_0)$ under the action of $\pi_1(X, x_0)$ is exactly $P_*\pi_1(\tilde{X}, y_0)$.

Proof. An element $[\sigma] \in \pi_1(X, x_0)$ satisfies $y_0 \cdot [\sigma] = y_0$, if and only if $\tilde{\sigma}_{y_0}$ is a loop. However, by Proposition 2.36, $\tilde{\sigma}_{y_0}$ is a loop if and only if $[\sigma] \in \pi_1(\tilde{X}, y_0)$. ☐

Proposition 2.40. Let $\tilde{X}$ be path connected, and let $x_0 \in X$ and $y_0 \in p^{-1}(x_0)$. Then the quotient set $\pi_1(X, x_0)/P_*\pi_1(\tilde{X}, y_0)$ and $p^{-1}(x_0)$ are bijective.

Proof. We leave this as an exercise to the reader. (Hint: Apply the Orbit-Stabilizer Theorem, which can be found in Theorem 17.2 in [Armstrong], and utilize some of the past propositions.) ☐

Now we can examine the Galois group of a covering space. First, let us define a few terms. Also, from now on, let's assume that the base space in a covering is path-connected. Note our following definitions:

Definition 2.41. A *morphism* from $(\tilde{X}_1, p_1)$ to $(\tilde{X}_2, p_2)$ is a continuous map $h : \tilde{X}_1 \to \tilde{X}_2$ such that $p_2 \circ h = p_1$. The morphism h is called an *isomorphism* if and only if it is also a homeomorphism as a map $h : \tilde{X}_1 \to \tilde{X}_2$. An *automorphism* of a covering space $\tilde{X}$ is a self-isomorphism $\phi : (\tilde{X}, p) \to (\tilde{X}, p)$ of the covering space.

Definition 2.42. The automorphisms of the covering space form a group under composition, which we call the *Galois group*, denoted as $Gal(\tilde{X}/X)$. We say that such covering space is *regular*, if and only if $Gal(\tilde{X}/X)$ acts transitively on the fiber $p^{-1}(x_0)$ for every $x_0 \in X$. A path connected covering space is *normal*, if and only if $P_*\pi_1(\tilde{X}, \tilde{x}_0)$ is a normal subgroup of $\pi_1(X, x_0)$, for every $x_0 \in X$ and every $\tilde{x}_0 \in p^{-1}(x_0)$. When the cover $\tilde{X}$ is simply connected, we say that such cover is *universal*.

Note also that the morphism h is also a covering map.

Example 2.43. Note that Figure 3 is regular, whereas Figure 4 is not.

Corollary 2.44. The number of sheets equals the index of $P_*\pi_1(\tilde{X}, y_0)$ in $\pi_1(X, x_0)$ for every $x_0 \in X$ and every $y_0 \in p^{-1}(x_0)$. If the covering is normal, then the number of sheets is equal to the size of the quotient group $\pi_1(X, x_0)/P_*\pi_1(\tilde{X}, y_0)$. If the cover is universal, then $\pi_1(X, x_0)$ is isomorphic to $p^{-1}(x_0)$ as sets.

Proof. Clear from Proposition 2.40. ☐

Proposition 2.45. Let $(\tilde{X}_1, p_1), (\tilde{X}_2, p_2)$ be two coverings over X. Let $x_0 \in X$, and let $y_1 \in p_1^{-1}(x_0), y_2 \in p_2^{-1}(x_0)$. Then there is a covering space isomorphism $h : (\tilde{X}_1, p_1) \to (\tilde{X}_2, p_2)$, if and only if $(P_1)_*\pi_1(\tilde{X}_1, y_1) = (P_2)_*\pi_1(\tilde{X}_2, y_2)$.

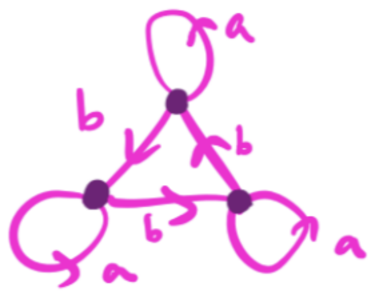

Figure 3. Example of a regular graph.

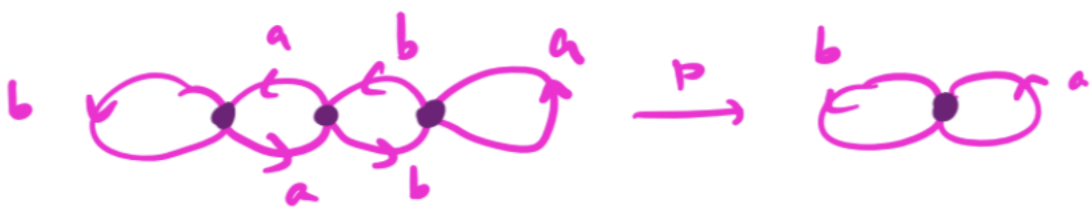

Figure 4. Example of an irregular graph.

Proof. The forward implication is clear: h must admit isomorphic fundamental groups in the covering space, which map to identical ones in the base space. The backward implication is left as a good exercise to the reader. (Hint: See the previous lemmas.) □

Now we state an important theorem on the Galois group of a covering space:

Theorem 2.46. Let $x_0 \in X$, $y_0 \in p^{-1}(x_0)$, and write G for $\pi_1(X, x_0)$, and H for $P_*\pi_1(\tilde{X}, y_0)$. Then the following hold:

(1) $Gal(\tilde{X}/X)$ is isomorphic to $N_G(H)/H$;
(2) $p : \tilde{X} \to X$ is normal if and only if it is regular.

Proof.

(1) Define $\Phi : Gal(\tilde{X}/X) \to N_G(H)/H$ as follows: Given $\varphi \in Gal(\tilde{X}/X)$, pick a path γ in $\tilde{X}$ from y_0 to $y_1 := \varphi(y_0)$. Write H' for $P_*\pi_1(\tilde{X}, y_1)$. We have $H' = [p \cdot \gamma]^{-1} H [p \cdot \gamma]$. So $[p \cdot \gamma] \in N_G(H)$. We define $\Phi(\varphi) = H[p \cdot \gamma]$. We leave it as an exercise to complete the rest of the proof (that is, show that Φ is indeed an isomorphism).
(2) We prove the forward and backward implications:
 ($\Rightarrow$) Let $y_0, y_1 \in p^{-1}(x_0)$. Since the covering is normal, $P_*\pi_1(\tilde{X}, y_0)$ is a normal subgroup of $\pi_1(X, x_0)$, then $P_*\pi_1(\tilde{X}, y_0) = P_*\pi_1(\tilde{X}, y_1)$ implies that there exists a $\varphi \in Gal(\tilde{X}/X)$ sending y_0 to y_1.
 ($\Leftarrow$) We need to show that for every $x_0 \in X$ and every $y_0 \in p^{-1}(x_0)$, $P_*\pi_1(\tilde{X}, y_0)$ is normal in $\pi_1(X, x_0)$. Let $[\sigma] \in \pi_1(X, x_0)$. Write y_1 for $\tilde{\sigma}_{y_0}(1)$. Given that $p : \tilde{X} \to X$ is regular, there is some $\varphi \in Gal(\tilde{X}/X)$ such that $\varphi(y_0) = y_1$. Now this implies that $P_*\pi_1(\tilde{X}, y_0) = P_*\pi_1(\tilde{X}, y_1)$ (i.e., $[\sigma]$ normalizes $P_*\pi_1(\tilde{X}, y_0)$). □

Corollary 2.47. The following results from Theorem 2.46:

(1) If $\tilde{X} \to X$ is normal, then $Gal(\tilde{X}/X)$ is isomorphic to $\pi_1(X, x_0)/P_*\pi_1(\tilde{X}, y_0)$.

(2) If $\tilde{X} \to X$ is universal, then $Gal(\tilde{X}/X)$ is isomorphic to $\pi_1(X, x_0)$.

Armed with the Galois theory of covering spaces, we are prepared to unfold:

3. Graph-theoretic coverings and zeta functions

We first fix a few terms and regarding finite graphs, following the convention of [Ter10]. The data of a *graph* X consists of a set V_X of vertices and a set E_X of edges, and the graph is *finite* if both V_X and E_X are finite sets. The *degree* of a vertex v is the number of edges connected to it, which we write as $\deg(v)$. The graphs we discuss are allowed to have loops and multiple edges, but we assume they do not contain degree-one vertices. A graph is *directed* if each of its edges is oriented.

Suppose that X is a finite graph with n vertices and m edges. One can (arbitrarily) orient the edges and label them by $e_1, \cdots, e_m$. We then write $e_{m+1} = e_1^{-1}, \cdots, e_{2m} = e_m^{-1}$ for the oppositely oriented edges. A *path* P in a graph X is a sequence $\{a_1, \cdots, a_k\}$ of edges (that is, $a_i \in \{e_1, \cdots, e_{2m}\}$ for each i), so that the end vertex of a_i is the start vertex of a_{i+1}, for $i = 1, \cdots, k-1$. In this case we write $P = a_1 a_2 \cdots a_k$. Such a path is *closed*, if the end vertex of a_k is the start vertex of a_1. Closed paths are also known as *loops*. The *length* of $P = a_1 a_2 \cdots a_k$ is k, which we denote by $\nu(P)$.

Definition 3.1. Let $P = a_1 a_2 \cdots a_k$ be a path in a finite graph X. We say that P is *backtrackless*, if a_{i+1} is not a_i^{-1} for every $i = 1, \cdots, k-1$. Say that P is *tailless*, if a_k is not a_1^{-1}. Paths differ by a cyclic permutation are called *equivalent*, so that

$$P = a_1 a_2 \cdots a_k \sim a_2 a_3 \cdots a_1 \sim \cdots \sim a_k a_1 \cdots a_{k-1}.$$

The equivalence class of P is denoted by $[P]$. A path P is *primitive*, if it is not of the form L^N with $N \geq 2$, where L is some loop in the graph and L^N denotes repeating L exactly N times.

Definition 3.2. A path P in a finite graph X is called a *prime*, if it is a backtrackless, tailless, primitive loop.

We are now in the position of defining the (vertex version of the) Ihara zeta function.

Definition 3.3 (Graph-theoretic zeta function)**.** The Ihara zeta function of a finite graph X is defined (for $u \in \mathbb{C}$ with $|u|$ small) as

$$\zeta_X(u) = \prod_{[P]} (1 - u^{\nu(P)})^{-1},$$

where the product is taken over all equivalence classes of primes in the graph X.

Example 3.4. Suppose that $X = C_n$ is a cyclic graph with n vertices. Then there are exactly two different primes in X up to equivalence (namely the clockwise loop and the anti-clockwise one), each of which has length n. It follows that

$$\zeta_{C_n}(u) = (1 - u^n)^{-2}.$$

So cyclic graph have quite simple zeta functions. A general non-cyclic graph X (e.g., the figure-eight graph) would contain infinitely many equivalence classes of primes, making the defining product of $\zeta_X(u)$ an infinite product. What makes zeta function much more computable is our main Theorem 1.3, a determinant formula proved independently by Hashimoto [Has89] and Bass [Bas92], which we repeat here.

Theorem 3.5 (See Theorem 1.3)**.** Let X be a finite graph with rank r_X (i.e., r_X is the rank of the fundamental group of X so that $r_X - 1 = |E_X| - |V_X| = m - n$). Then

$$\zeta_X(u)^{-1} = (1 - u^2)^{r_X - 1} \det(I_X - A_X u + Q_X u^2).$$

Some definitions are in order. The matrix I_X is the identity matrix of size $n = |V_X|$. The matrix Q_X is a diagonal matrix of the same size, whose (i,i)-entry equals $\deg(v_i) - 1$, for $i = 1, \cdots, n$. The matrix A_X is known as the *adjacency matrix* of X defined as follows:

Definition 3.6. Let X be a finite graph with n vertices $v_1, \cdots, v_n$. Its adjacency matrix A_X is a square matrix of size n, whose (i,j)-entry is

$$a_{ij} = \begin{cases} \text{the number of (undirected) edges connecting } v_i \text{ and } v_j, & \text{if } i \neq j; \\ \text{twice the number of loops at } v_i, & \text{if } i = j. \end{cases}$$

We will prove this main theorem in Section 4, but let us present some examples here.

Example 3.7. Consider the tetrahedron graph T pictured below.

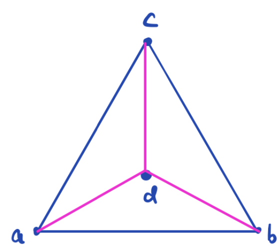

FIGURE 5. The tetrahedron graph T.

The adjacency matrix of the tetrahedron graph is

$$A_T = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

and therefore

$$\begin{aligned} \zeta_T(u)^{-1} &= (1-u^2)^2 \det \begin{bmatrix} 1+2u^2 & -u & -u & -u \\ -u & 1+2u^2 & -u & -u \\ -u & -u & 1+2u^2 & -u \\ -u & -u & -u & 1+2u^2 \end{bmatrix} \\ &= (1-u^2)^2(1-u)(1-2u)(1+u+2u^2)^3. \end{aligned}$$

Example 3.8. Consider the graph Y pictured below.
The adjacency matrix of this graph equals

$$A_Y = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$

and therefore

$$\begin{aligned} \zeta_Y(u)^{-1} &= (1-u^2)^3 \det \begin{bmatrix} 1-2u+3u^2 & -u & -u \\ -u & 1-2u+3u^2 & -u \\ -u & -u & 1-2u+3u^2 \end{bmatrix} \\ &= (1-u^2)^3(1-4u+3u^2)(1-u+3u^2)^2. \end{aligned}$$

Finite graphs support a nice theory of fundamental groups and covering spaces. Using knowledge from Section 2 one proves immediately the following basic results:

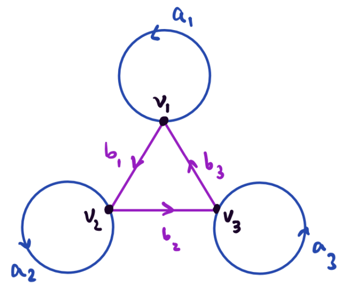

FIGURE 6. The graph Y.

Proposition 3.9. Every connected graph X contains a maximal tree T, and the fundamental group $\pi_1(X)$ is a free group with generators correspond exactly to edges not in T.

Proposition 3.10. Every covering space of a graph is again a graph, with vertices and edges in the covering graph lifting those in the base graph.

Detailed proofs of the above results can be found in [Hat02, Section 1.A]. A remark here is that these results can be applied to prove that every subgroup of a free group is free – a celebrated result in group theory that is not so easy to prove directly in algebra.

Since graphs come equipped with the data of vertices and edges, they are slightly more than their underlying topological spaces. We now spell out the definition of a unramified finite covering of graphs. Assume that graphs X, Y are finite and connected. We say that the graph Y is an *unramified covering* of the graph X, if there is a covering map $p : Y \to X$ which takes adjacent vertices in Y to adjacent vertices in X, such that for every $x \in X$ and every $y \in p^{-1}(x)$, points adjacent to y in Y are mapped bijectively onto points adjacent to x in X. An n-sheeted unramified covering $p : Y \to X$ is *normal*, or *Galois*, if there are exactly n automorphisms of Y over X. (That is to say that $p : Y \to X$ is a normal covering space in the usual sense.) The group of automorphisms will be denoted by $\mathrm{Gal}(Y/X)$ as usual.

Example 3.11. Below is a two 2-sheeted covering of the tetrahedron graph which is different from that in Example 1.1.

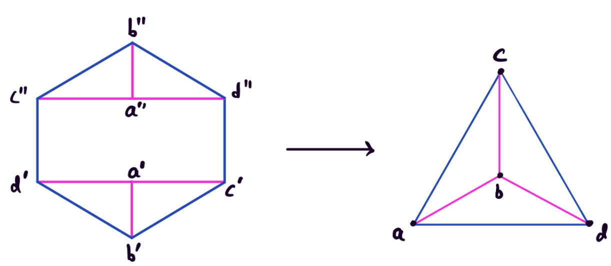

FIGURE 7. Another 2-sheeted cover of the figure-eight graph.

The Galois theory of covering spaces can be specialized in the context of coverings of graphs, so that there is a bijective correspondence between intermediate graphs of a given graph covering $Y \to X$ and subgroups of the automorphism group $\mathrm{Gal}(Y/X)$. In fact, we have the following:

Theorem 3.12 (Fundamental theorem of Galois coverings of graphs)**.** Suppose that $Y \to X$ is an unramified normal covering of finite graphs, with Galois group $G = \mathrm{Gal}(Y/X)$. Then:

(1) Given any intermediate graph $\tilde{X}$, the covering $Y \to \tilde{X}$ is normal, with $\mathrm{Gal}(Y/\tilde{X})$ a subgroup of G.
(2) Conversely, given any subgroup H of the Galois group G, there is an intermediate graph $\tilde{X}$ with $\mathrm{Gal}(Y/\tilde{X})$ exactly equals H.
(3) Two intermediate graphs $\tilde{X}$ and $\tilde{X}'$ are isomorphic precisely when $\mathrm{Gal}(Y/\tilde{X})$ equals $\mathrm{Gal}(Y/\tilde{X}')$.
(4) Two intermediate graphs $\tilde{X}$, $\tilde{X}'$ satisfy that $\tilde{X}$ covers $\tilde{X}'$ precisely when $\mathrm{Gal}(Y/\tilde{X})$ is a subgroup of $\mathrm{Gal}(Y/\tilde{X}')$.
(5) An intermediate graph $\tilde{X}$ is normal over X precisely when $\mathrm{Gal}(Y/\tilde{X})$ is a normal subgroup of $\mathrm{Gal}(Y/X)$, in which case $\mathrm{Gal}(\tilde{X}/X)$ is isomorphic to the quotient group $\mathrm{Gal}(Y/X)/\,\mathrm{Gal}(Y/\tilde{X})$.

A detailed proof of the above theorem, together with more discussions on coverings of graphs, can be found in [Ter10, Section 14]. For us, the most important feature of finite graph coverings is the way they interact with zeta functions of graphs. Another main goal of the paper is to show that a finite graph covering gives rise to a division of the inverse of the zeta functions.

Theorem 3.13 (See Theorem 1.5)**.** Suppose that $Y \to X$ is a finite covering of finite connected graphs (not necessarily normal), then $\zeta_Y(u)^{-1}$ divides $\zeta_X(u)^{-1}$.

This is our second main result which we will prove in Section 4. An illustrating example has already been given in Example 1.6. Let us now present a few more examples.

Example 3.14. Let X be the figure-eight space, and let Y be the graph as in Example 3.8. Then Y covers X as indicated by the following picture.

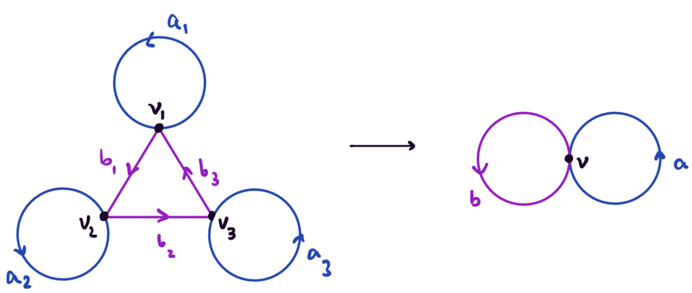

FIGURE 8. A 3-sheeted cover of the figure-eight graph.

The zeta function of the figure-eight graph has inverse

$$\zeta_X(u)^{-1} = (1-u^2)(1-4u+3u^2),$$

while the covering graph Y has

$$\zeta_Y(u)^{-1} = (1-u^2)^3(1-4u+3u^2)(1-u+3u^2)^2$$

as computed in Example 3.8. It is clear that $\zeta_Y(u)$ is divisible by $\zeta_X(u)$.

Example 3.15. Let $Y \to X$ be the following graph covering, with $\tilde{X}$ as an intermediate graph.

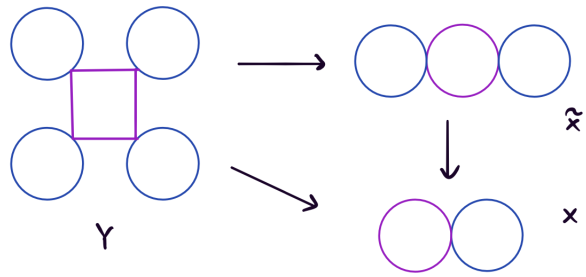

Figure 9. An intermediate graph of a graph covering.

The inverse of the zeta function of the spaces involved are:

$$\begin{aligned}
\zeta_X(u)^{-1} &= (1-u^2)(1-4u+3u^2),\\
\zeta_{\tilde{X}}(u)^{-1} &= (1-u^2)^2(1+3u^2)(1-4u+3u^2),\\
\zeta_Y(u)^{-1} &= (1-u^2)^4(1-4u+3u^2)(1+3u^2)(1-2u+3u^2).
\end{aligned}$$

It is clear that $\zeta_{\tilde{X}}(u)^{-1}$ divides $\zeta_{\tilde{X}}(u)^{-1}$, and that $\zeta_{\tilde{X}}(u)^{-1}$ divides $\zeta_Y(u)^{-1}$.

4. Proof of the main theorems

We now present proofs of our main results, namely Theorem 1.3 and Theorem 1.5.

4.1. **Proof of Theorem 1.3.** Our main theorem is the determinant formula for the Ihara zeta function:

$$\zeta(u,X) = (1-u^2)^{r_X-1}\cdot \det(I-uA_X+u^2Q_X)^{-1},$$

where $r_X-1=|E|-|V|$, A_X is the adjacency matrix of the graph X, and Q_X is the diagonal matrix whose entries are the degrees of the vertices in X. We follow the proof strategy developed by Horton, Stark, and Terras.

Strategy. The proof proceeds in two main stages. First, we define the *edge zeta function* $\zeta_E(W,X)$ and prove that it equals $\det(I-W)^{-1}$, where W is the non-backtracking edge matrix. This is done by applying the Euler differential operator to the logarithm of the product expansion.

Second, we specialize the weights in W, and use a matrix factorization involving the flip matrix J, start matrix S, and end matrix T to relate $\det(I-uW_1)$ to $\det(I-uA_X+u^2Q_X)$, thus recovering the Ihara determinant formula.

Definition: Edge Zeta Function. The edge zeta function $\zeta_E(W,X)$ is defined as

$$\zeta_E(W,X) = \prod_{[P]}\left(1-N(P)\right)^{-1},$$

where the product runs over equivalence classes of primitive, backtrackless closed cycles $[P]$ in the directed edge graph of X. Here, $N(P)$ is the *edge norm* of P, defined as the product of weights w_e assigned to each directed edge e appearing in the cycle P.

These weights are arranged in the $2|E|\times 2|E|$ non-backtracking edge matrix W, whose rows and columns are indexed by the directed edges $e_1,\dots,e_{2|E|}$. The entry W_{e_i,e_j} is w_{e_j} if the terminal vertex of e_i equals the initial vertex of e_j and $e_j\neq \bar{e}_i$ (i.e., no backtracking), and 0 otherwise.

Specialization to Ihara Zeta Function. When we specialize all nonzero weights w_e to be a fixed variable u, the edge norm becomes

$$N(P) = u^{\ell(P)},$$

where $\ell(P)$ is the length of the cycle P. This is sometimes denoted $v(P)$ in other sources. Let W_1 be the matrix W with all nonzero entries set to 1. Then, we define the Ihara zeta function as

$$\zeta(u, X) := \zeta_E(uW_1, X).$$

Intermediate Theorem. With these definitions, our first major step is to prove the following result:

Edge Zeta Function Determinant Formula

$$\zeta_E(W, X) = \det(I - W)^{-1}.$$

We now prove this identity by analyzing the logarithm of the product expansion of $\zeta_E(W, X)$.
Step 1: Take logarithms. Taking the natural logarithm of both sides of the Euler product:

$$\ln \zeta_E(W, X) = \ln \prod_{[P]} (1 - N(P))^{-1}.$$

By properties of logarithms:

$$\ln \zeta_E(W, X) = -\sum_{[P]} \ln (1 - N(P)).$$

Step 2: Expand using Taylor series. Recall the identity:

$$\ln(1 - x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}, \quad \text{for } |x| < 1.$$

Applying this to each $N(P)$, we obtain:

$$\ln \zeta_E(W, X) = \sum_{[P]} \sum_{k=1}^{\infty} \frac{N(P)^k}{k}.$$

Step 2.5: Expand using the Euler differential operator. To re-express this sum more structurally, we apply the Euler differential operator:

$$L := \sum_{i,j} w_{ij} \frac{\partial}{\partial w_{ij}}.$$

This operator acts on a monomial $w_{i_1 j_1} \cdots w_{i_t j_t}$ by returning its total degree:

$$L(w_{i_1 j_1} \cdots w_{i_t j_t}) = t \cdot w_{i_1 j_1} \cdots w_{i_t j_t}.$$

Since each $N(P)^k$ is a monomial of degree $k \cdot \ell(P)$, applying L gives:

$$L \ln \zeta_E(W, X) = \sum_{[P]} \sum_{k=1}^{\infty} N(P)^k = \sum_{C} N(C),$$

where the sum is now over all backtrackless closed walks C, not necessarily primitive. Each such walk arises uniquely as a k-fold repetition of a primitive cycle.

Step 3: Interpret as a trace. The quantity $\sum_C N(C)$ equals the sum of traces of powers of the edge matrix:

$$\sum_C N(C) = \sum_{m=1}^{\infty} \operatorname{Tr}(W^m),$$

and so

$$L \ln \zeta_E(W, X) = \sum_{m=1}^{\infty} \operatorname{Tr}(W^m).$$

Step 4: Apply L to the determinant. We also have:

$$\ln \det(I - W)^{-1} = \sum_{m=1}^{\infty} \frac{1}{m} \operatorname{Tr}(W^m),$$

and using the fact that L applied to $\frac{1}{m}\operatorname{Tr}(W^m)$ gives $\operatorname{Tr}(W^m)$, we get:

$$L \ln \det(I - W)^{-1} = \sum_{m=1}^{\infty} \operatorname{Tr}(W^m).$$

Step 5: Conclude the determinant identity. We now conclude:

$$L \ln \zeta_E(W, X) = L \ln \det(I - W)^{-1}.$$

Since both sides agree under L, and both vanish when all weights are set to zero, it follows that

$$\zeta_E(W, X) = \det(I - W)^{-1}.$$

We now complete the proof by specializing the matrix W to recover the Ihara zeta function. Step 6: Specialize $W = uW_1$ and define the Ihara zeta function. Let W_1 denote the unweighted non-backtracking edge matrix of X, with entries in $\{0, 1\}$, so that all nonzero weights are replaced by 1. Set:

$$W := uW_1.$$

Then we define the Ihara zeta function as the specialization:

$$\zeta(u, X) := \zeta_E(uW_1, X).$$

By the Edge Zeta Determinate Formula, we have:

$$\zeta(u, X) = \det(I - uW_1)^{-1}.$$

Step 7: Return to the main theorem. We can now return to complete the proof of the main theorem.

Let the number of edges in the graph be $m = |E|$, so that there are $2m$ directed edges, and let the number of vertices be $n = |V|$.

We now define several matrices that relate directed edges and vertices:

- The *flip matrix* $J \in \mathbb{R}^{2m \times 2m}$ is the permutation matrix that swaps each directed edge with its reverse:

$$J_{e,e'} = \begin{cases} 1 & \text{if } e' = \bar{e}, \\ 0 & \text{otherwise.} \end{cases}$$

- The *start matrix* $S \in \mathbb{R}^{n \times 2m}$ records which vertex is the source of a directed edge:

$$S_{v,e} = \begin{cases} 1 & \text{if } o(e) = v, \\ 0 & \text{otherwise.} \end{cases}$$

- The *end matrix* $T \in \mathbb{R}^{n \times 2m}$ records which vertex is the target of a directed edge:

$$T_{v,e} = \begin{cases} 1 & \text{if } t(e) = v, \\ 0 & \text{otherwise.} \end{cases}$$

Step 8: Matrix identities. These matrices satisfy the following key identities:

(1) $SJ = T$ and $TJ = S$,
(2) $S^\top S = T^\top T = Q_X + I$, where Q_X is the degree matrix,
(3) $T^\top S = W_1 + J$, where W_1 is the unweighted non-backtracking edge matrix.

These facts allow us to write the determinant of $I - uW_1$ in terms of matrices defined over the vertex set.

Step 9: Apply the determinate identity. From a matrix factorization identity due to Bass and refined by Horton, Stark, and Terras, we have:

$$\det(I - uW_1) = (1 - u^2)^{r_X - 1} \cdot \det(I - uA_X + u^2 Q_X),$$

where:

- A_X is the adjacency matrix of the graph X,
- Q_X is the diagonal matrix of vertex degrees,
- $r_X = m - n + 1$ is the cyclomatic number of X.

Step 10: Conclude. Substituting into our earlier formula:

$$\zeta(u, X) = \zeta_E(uW_1, X) = \det(I - uW_1)^{-1},$$

we obtain:

$$\zeta(u, X) = \left[(1 - u^2)^{r_X - 1} \cdot \det(I - uA_X + u^2 Q_X)\right]^{-1},$$

and thus:

$$\zeta(u, X) = (1 - u^2)^{r_X - 1} \cdot \det(I - uA_X + u^2 Q_X)^{-1},$$

which proves the main theorem. □

4.2. **Proof of Theorem 1.5.**

Proof. Begin by applying Theorem 1.3 so that $\zeta_X(u)^{-1} = (1-u^2)^{r_x - 1} \det(I_X - A_X u + Q_X u^2)$ and $\zeta_Y(u)^{-1} = (1-u^2)^{r_y - 1} \det(I_Y - A_Y u + Q_Y u^2)$.

It will suffice to show that $(1-u^2)^{r_x-1}$ divides $(1-u^2)^{r_y-1}$ and that $\det(I_X - A_X u + Q_X u^2)$ divides $\det(I_Y - A_Y u + Q_Y u^2)$.

That $(1-u^2)^{r_x-1}$ divides $(1-u^2)^{r_y-1}$ follows immediately from the fact that p is a finite covering. Namely, we know then that p is n-sheeted for some $n < \infty$ and so $r_y - 1 = n(r_x - 1)$ (since $r_y - 1 =$ number of edges - number of vertices).

To show that $\det(I_X - A_X u + Q_X u^2)$ divides $\det(I_Y - A_Y u + Q_Y u^2)$, order the vertices of Y into blocks corresponding to the sheets of of the cover. Now, A_Y will be made up of blocks A_{ij} where $1 \leq i, j \leq n$. $\Sigma_j A_{ij} = A$ should be clear from unique path lifting Lemma 2.29.

With the same block ordering of Y, Q_Y will be composed of n copies of Q_X along the diagonal and so too will I_Y be n copies of I_X along the diagonal.

Now, the $n-1$ block columns of $I_Y - A_Y u + Q_Y u^2$ can be added to the first block column without changing the determinant. This result is a matrix with n copies of $I_X - A_X u + Q_X u^2$ in the first block column,

$$\begin{bmatrix} I_X - A_X u + Q_X u^2 & \dots \\ I_X - A_X u + Q_X u^2 & \dots \\ \dots & \dots \\ I_X - A_X u + Q_X u^2 & \dots \end{bmatrix} \tag{4}$$

Additionally, the first block row can be subtracted from every other row without changing the determinant. This results in zeroes in all the rows of the first block column except for top left block. So, $I_Y - A_Y u + Q_Y u^2$ is row equivalent to,

$$\begin{bmatrix} I_X - A_X u + Q_X u^2 & ... \\ 0 & ... \\ ... & ... \\ 0 & ... \end{bmatrix} \tag{5}$$

Naturally, it follows that $\det(I_X - A_X u + Q_X u^2)$ divides $\det(I_Y - A_Y u + Q_Y u^2)$.

□

Appendix A. Solution of the Basel Problem

We will prove that $\zeta(2) = \frac{\pi^2}{6}$ using integral calculus. Let n be an integer with $n \geq 0$. We define, for each such $n \geq 0$,

$$A_n = \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx \tag{6}$$

$$B_n = \int_0^{\frac{\pi}{2}} x^2 \cos^{2n} x \, dx. \tag{7}$$

Proposition A.1. For all $n \geq 1$,

$$nA_n = \frac{2n-1}{2} A_{n-1}. \tag{8}$$

Proof. In (6), let $u = \cos^{2n-1} x$ and $dv = \cos x$. After applying integration by parts through this substitution onto A_n, we obtain

$$\begin{aligned} A_n &= (2n-1) \left[\int_0^{\frac{\pi}{2}} \cos^{2n-2} x \, dx - \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx \right] \\ &= (2n-1) \left[A_{n-1} - A_n \right] \end{aligned}$$

which, after some algebraic manipulation, results in (8). □

Proposition A.2. For all $n \geq 1$,

$$A_n = 2n \int_0^{\frac{\pi}{2}} x \sin x \cos^{2n-1} x \, dx. \tag{9}$$

Proof. Applying integration by parts on (6), we obtain

$$\begin{aligned} A_n &= x \cos^{2n}(x) |_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} x \, d(\cos^{2n} x) \\ &= 0 - \int_0^{\frac{\pi}{2}} x \cdot 2n \cdot \cos^{2n-1}(x)(-\sin(x)) \, dx \\ &= 2n \int_0^{\frac{\pi}{2}} x \sin(x) \cos^{2n-1}(x) \, dx, \end{aligned}$$

as desired. □

Proposition A.3. For all $n \geq 1$,

$$\frac{A_n}{n^2} = \frac{2n-1}{n} B_{n-1} - 2B_n. \tag{10}$$

Proof. It follows from (9) that

$$\frac{A_n}{2n} = \int_0^{\frac{\pi}{2}} x \sin x \cos^{2n-1} x \, dx.$$

Applying integration by parts on the above integral, we obtain

$$\begin{aligned}
\frac{A_n}{2n} &= \int_0^{\frac{\pi}{2}} \sin x \cos^{2n-1} x \, d(\frac{1}{2}x^2) \\
&= \frac{1}{2}x^2 \sin(x) \cos^{2n-1}(x)|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{1}{2}x^2 \left[\cos^{2n}(x) - (2n-1)\sin^2(x)\cos^{2n-2}(x)\right] \, dx \\
&= -\int_0^{\frac{\pi}{2}} \frac{1}{2}x^2 \cos^{2n}(x) \, dx + \int_0^{\frac{\pi}{2}} \frac{2n-1}{2} x^2 \left(1 - \cos^2(x)\right) \cos^{2n-2}(x) \, dx \\
&= -\frac{1}{2}B_n - \frac{2n-1}{2}B_n + \frac{2n-1}{2}B_{n-1} \\
&= \frac{2n-1}{2}B_{n-1} - nB_n,
\end{aligned}$$

and (10) follows immediately. □

Proposition A.4. For all $n \geq 1$,

$$\frac{1}{n^2} = 2(\frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n}). \tag{11}$$

Proof. First, we re-write (10) as

$$\frac{1}{n^2} = \frac{2n-1}{n} \frac{B_{n-1}}{A_n} - 2\frac{B_n}{A_n}.$$

Next we note that, by (8),

$$\frac{1}{A_n} = \frac{2n}{2n-1} \cdot \frac{1}{A_{n-1}}$$

It follows that

$$\begin{aligned}
\frac{1}{n^2} &= \frac{2n-1}{n} \frac{B_{n-1}}{A_n} - 2\frac{B_n}{A_n} \\
&= \frac{2n-1}{n} B_{n-1} \cdot \frac{2n}{2n-1} \cdot \frac{1}{A_{n-1}} - 2\frac{B_n}{A_n} \\
&= 2\frac{B_{n-1}}{A_{n-1}} - 2\frac{B_n}{A_n}.
\end{aligned}$$

□

Proposition A.5. For all $n \geq 1$,

$$\sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} - 2\frac{B_n}{A_n}. \tag{12}$$

Proof. Taking $n = k$ in (11) and adding such equations together for $k = 1, \cdots, n$, one obtains that

$$\sum_{k=1}^{n} \frac{1}{k^2} = 2\left(\frac{B_0}{A_0} - \frac{B_n}{A_n}\right)$$

Since

$$A_0 = \int_0^{\frac{\pi}{2}} 1 \cdot dx = \frac{\pi}{2}, \quad B_0 = \int_0^{\frac{\pi}{2}} x^2\, dx = \frac{\pi^3}{24},$$

It follows that

$$\sum_{k=1}^{n} \frac{1}{k^2} = 2\left(\frac{B_0}{A_0} - \frac{B_n}{A_n}\right) = \frac{\pi^2}{6} - 2\frac{B_n}{A_n}.$$

□

Proposition A.6. For all $n \geq 0$,

$$B_n \leq \int_0^{\frac{\pi}{2}} x^2 \left(1 - \frac{4x^2}{\pi^2}\right)^n dx. \tag{13}$$

Proof. We note that

$$B_n = \int_0^{\frac{\pi}{2}} x^2 \cos^{2n} x\, dx = \int_0^{\frac{\pi}{2}} x^2 \left(1 - \sin^2 x\right)^n\, dx.$$

Using the fact that $\sin(x) \geq \frac{2}{\pi}x$ whenever $0 \leq x \leq \frac{\pi}{2}$, we have

$$B_n = \int_0^{\frac{\pi}{2}} x^2 \left(1 - \sin^2 x\right)^n\, dx \leq \int_0^{\frac{\pi}{2}} x^2 \left(1 - \frac{4x^2}{\pi^2}\right)^n dx.$$

□

Proposition A.7. For all $n \geq 0$,

$$\int_0^{\frac{\pi}{2}} x^2 \left(1 - \frac{4x^2}{\pi^2}\right)^n dx = \frac{\pi^2}{8(n+1)} \int_0^{\frac{\pi}{2}} \left(1 - \frac{4x^2}{\pi^2}\right)^{n+1} dx. \tag{14}$$

Proof. On the left hand side of Equation (14), apply integration by parts using the substitution $u = x$ and $dv = x\left(1 - \frac{4x^2}{\pi^2}\right)^2$. The right hand side of the equation follows. □

Proposition A.8. For all $n \geq 0$,

$$B_n \leq \frac{\pi^3}{16(n+1)} \int_0^{\frac{\pi}{2}} \cos^{2n+3} t\, dt \leq \frac{\pi^3}{16(n+1)} A_n. \tag{15}$$

Proof. It follows from Equations (13) and (14) that

$$B_n \leq \frac{\pi^3}{16(n+1)} \int_0^{\frac{\pi}{2}} \cos^{2n+3} t\, dt \leq \frac{\pi^3}{16(n+1)} A_n. \tag{16}$$

Substituting $x = \frac{\pi}{2}\sin t$ into Equation (16) yields Equation (15). □

Proposition A.9. For every $n \geq 1$,

$$\frac{\pi^2}{6} - \frac{\pi^3}{8(n+1)} \leq \sum_{k=1}^{n} \frac{1}{k^2} < \frac{\pi^2}{6}. \tag{17}$$

Proof. The left-hand side of Equation (17) comes immediately from substituting the upper bound of $\frac{B_n}{A_n}$ obtained in Equation (15) into Equation (12). The right-hand side of Equation (17) comes from the fact that $\frac{B_n}{A_n} > 0$ (as Equations (6) and (7) must both be positive). □

Theorem A.10. We obtain

$$\zeta(2) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{18}$$

Proof. In Equation (17), note that $\sum_{k=1}^{n} \frac{1}{k^2}$ has $\frac{\pi^2}{6}$ as both an upper bound and lower bound as $n \to \infty$. Equation (18) results immediately from the squeeze theorem. $\square$

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