Welcome to REU in Topology

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Outline

1. Knowing our team
2. Learning our goals
3. Introducing our topics

Our team

- **Mark Benecke**
- **Mason Gardner**
- **Yang Hu** (He/him/his.)
  - Postdoctoral Fellow at NMSU.
  - Algebraic topologist and homotopy theorist.
- **Shokhina Jalilova**

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  - Analysis – the classical Riemann zeta function.

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  - ☐ Even better, create new mathematical knowledge!

Our central topic

Riemann Zeta Functions of Graphs.

Our focus today

What is the (classical) Riemann zeta function
and how do you define it for graphs?

Pop quiz

$$\sum_{k\geq 1}\frac{1}{k^2} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots+\frac{1}{n^2}+\cdots = ??$$

Pop quiz

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Answer: The infinite sum equals $\frac{\pi^2}{6}$.

History: the Basel problem

$$\sum_{k\geq 1} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} + \cdots = ??$$

Question: Does the Basel series converge?

If so, what does it converge to?

History: the Basel problem

The Basel series is an example of a *p*-series.

Definition

A *p*-series is a series of the form

$$\sum_{k \geq 1} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p} + \cdots$$

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Example

- The Basel series is a p-series when $p = 2$.
- The harmonic series is a p-series when $p = 1$.

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Theorem

The p-series converges when $p > 1$, and diverges when $p \leq 1$.

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In a few lucky cases, one can make precise computations with convergent series.

Example

The geometric series $\sum_{k\geq 1} ar^k = a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots$ is computable.

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More generally, the theory of power series (for which one can study the interval of convergence) produces interesting infinite sums, like:

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- $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\cdots = \frac{\pi}{4}.$

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- $1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots + \frac{1}{n^4} + \cdots = \frac{\pi^4}{90}$.

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Today, let's prove that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6},$$

just using knowledge from elementary calculus.

To sum up

So far, we know the following things about the p-series $\sum_{n \geq 1} \frac{1}{n^p}$:

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Open Problem

It is still an open problem to give a closed-form description of

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots + \frac{1}{n^3} + \cdots .$$

The naive Riemann zeta function

Let's now switch our perspective and notation. Replace "p" by "s" in the p-series:

$$\sum_{n\geq 1} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots + \frac{1}{n^s} + \cdots ,$$

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The (naive) Riemann zeta function is defined to be

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots + \frac{1}{n^s} + \cdots.$$

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At this point, the domain of $\zeta(s)$ consists of all real numbers s so that $s > 1$.

Time to switch gears

What are prime numbers

and how are they distributed?

History: the Prime Number Theorem

Definition

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Theorem

When x is large enough, $\pi(x)$ is roughly $\mathrm{Li}(x)$.

Connecting the two stories

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$$\prod_{p \text{ prime}} (1 - p^{-s})^{-1} = \frac{1}{1 - \frac{1}{2^s}} \times \frac{1}{1 - \frac{1}{3^s}} \times \frac{1}{1 - \frac{1}{5^s}} \times \frac{1}{1 - \frac{1}{7^s}} \times \frac{1}{1 - \frac{1}{11^s}} \times \cdots.$$

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The Riemann Hypothesis

All nontrivial zeros of $\zeta(s)$ are located along the line $\mathrm{Re}(s) = \frac{1}{2}$.

You win $1,000,000 if you solve the Riemann Hypothesis.

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Definition

Ihara The zeta function associated with a graph X is

$$\zeta(u, X) := \prod_{[P]} (1 - u^{\nu(P)})^{-1},$$

where $\nu(P)$ denotes the number of vertices in P.

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Let's end with a few examples today.